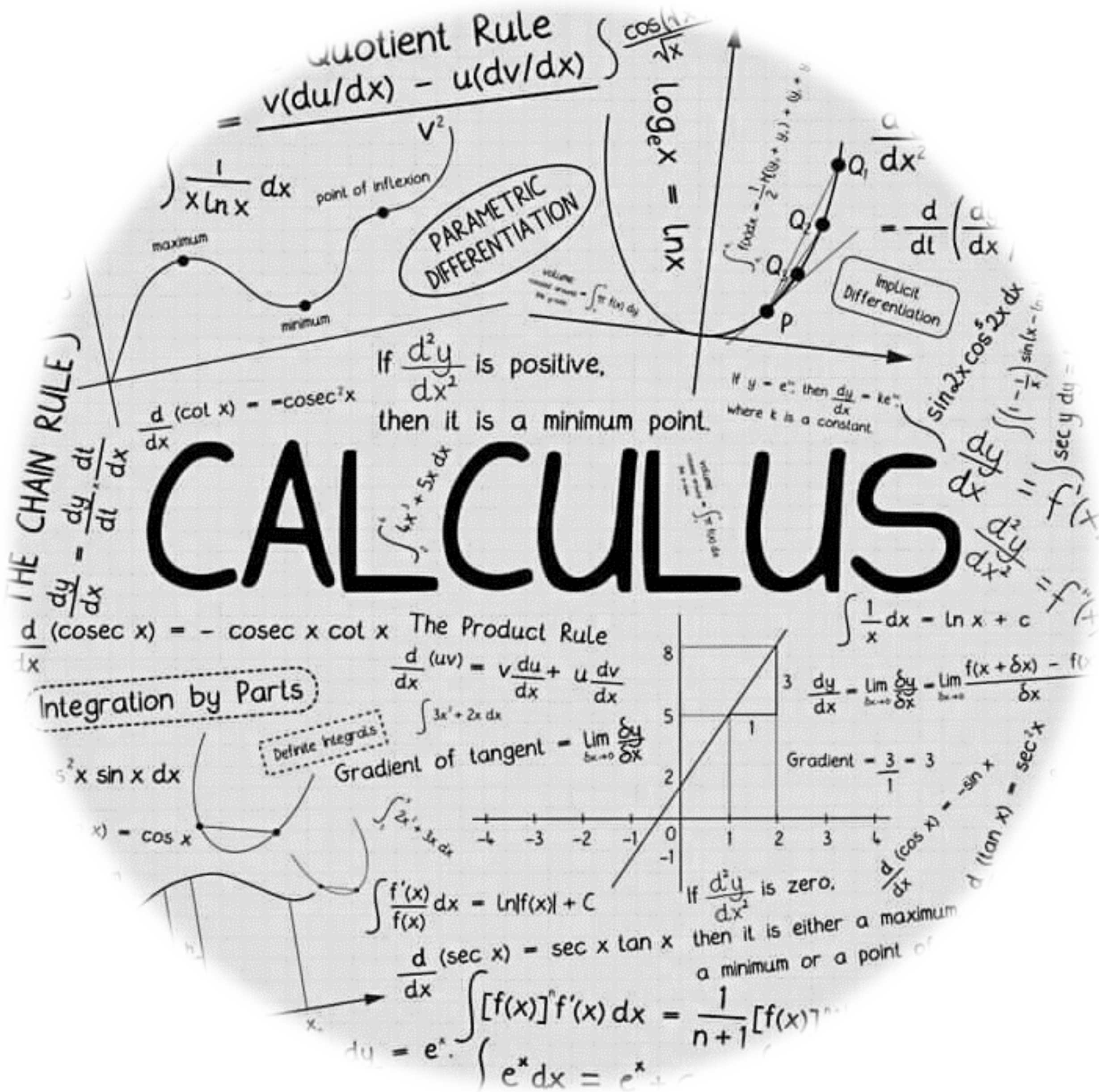
eSpyMath: AP Calculus AB Textbook

Contents

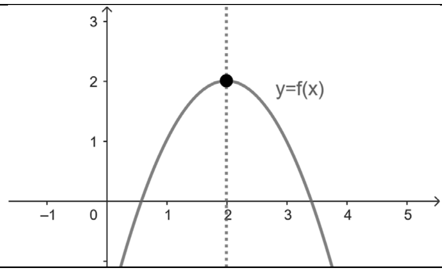
Chapter 0. Foundations and Previews

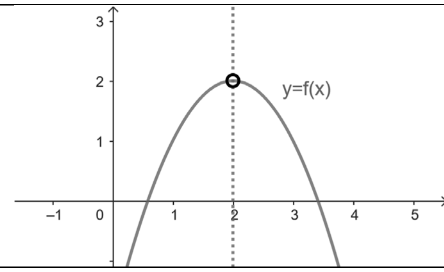


0-1. Why Calculus

1) Value of $f(x)$ when $x = c$:

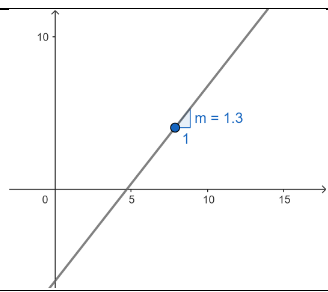
Without Calculus	With Differential Calculus
You directly find the value of the function f at $x = c$.	You consider the limit of $f(x)$ as x approaches c, which can be more precise, especially if f is not continuous at c.

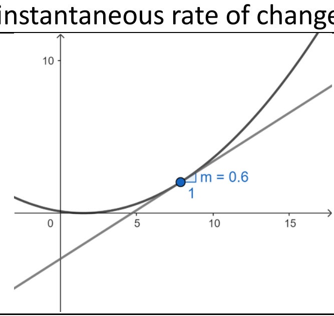




2) Slope of a line:

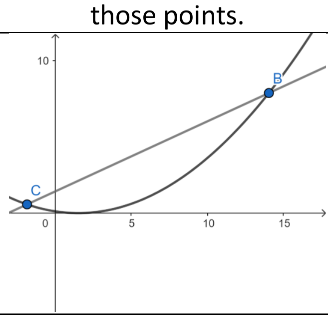
Without Calculus	With Differential Calculus
The slope is the change in y divided by the change in x ($\Delta y / \Delta x$).	**The slope of a curve at a point** is found using the derivative (dy / dx), representing the instantaneous rate of change.

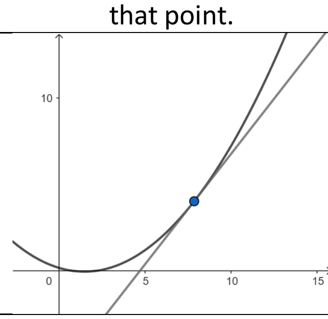




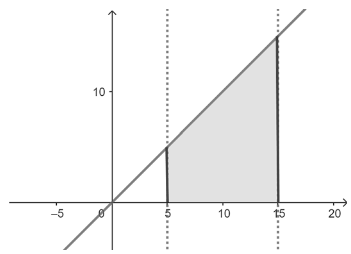
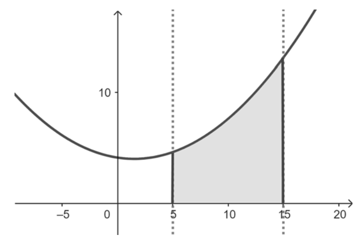
3) Secant line to a curve:

Without Calculus	With Differential Calculus
A secant line intersects the curve at two points, representing the average rate of change between those points.	**A tangent line** touches the curve at one point, representing the instantaneous rate of change at that point.





4) Area under the line or curve:

Without Calculus	With Differential Calculus
You find the area by multiplying the length and width of the rectangle/polygon.	**You find the area under a curve** using integration, which can handle more complex shapes.
	

5) Length of a line segment:

Without Calculus	With Differential Calculus
The length is the distance between two points.	You find the **length of an arc** (curved line) using integration.

6) Surface area of a cylinder:

Without Calculus	With Differential Calculus
You calculate the surface area using the formula for a cylinder.	You find **the surface area of a solid** of revolution using integration, which can handle more complex shapes.

7) Mass of a solid of constant density:

Without Calculus	With Differential Calculus
The mass is found by multiplying the volume by the constant density.	You calculate the **mass of a solid** with variable density using integration.

8) Volume of a rectangular solid:

Without Calculus	With Differential Calculus
The volume is found by multiplying length, width, and height.	You find **the volume of a region under a surface** using integration.

0-2. Increasing & Decreasing functions (use in curve sketching)

Examples:

1) Determine the domain and range of the function $g(x) = 2e^x$.

The function $g(x) = 2e^x$ is a transformation of the basic exponential function e^x.

- Domain: The domain of e^x is all real numbers, $(-\infty, \infty)$. Since $g(x) = 2e^x$ is just a vertical stretch, the domain remains the same.
- Domain: $(-\infty, \infty)$
- Range: The range of . e^x . is $(0, \infty)$. Multiplying by 2 stretches the range but does not change its lower or upper bounds.
- Range: $(0, \infty)$

2) Determine the domain and range of the function $g(x) = 2\sin(x)$.

The function $g(x) = 2\sin(x)$ is a vertical stretch of the basic sine function $\sin(x)$.

- Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty, \infty)$. The vertical stretch does not affect the domain.
- Domain: $(-\infty, \infty)$
- Range: The range of $\sin(x)$ is $[-1, 1]$. Multiplying by 2 stretches the range to $[-2, 2]$.
- Range: $[-2, 2]$

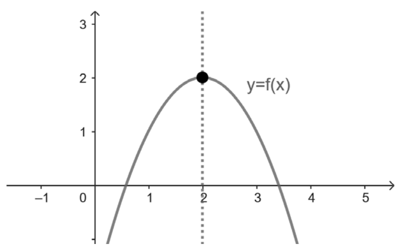
3) Determine the domain and range of the function $g(x)=\frac{1}{2x}$**.**

The function $g(x)=\frac{1}{2x}$ is a vertical compression of the basic function $\frac{1}{x}$.

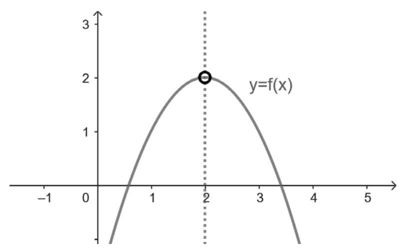
- Domain: The domain of $\frac{1}{x}$ is all real numbers except $x=0$, because division by zero is undefined. The vertical compression does not affect the domain.
- Domain: $(-\infty,0)\cup(0,\infty)$
- Range: The range of $\frac{1}{x}$ is all real numbers except $y=0$, since $\frac{1}{x}$ never equals zero. The vertical compression does not affect the range.
- Range: $(-\infty,0)\cup(0,\infty)$

0-3. Limit Foundation

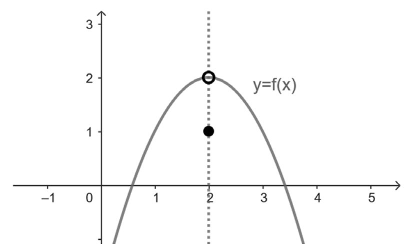
Examples: Find limits for the graph below



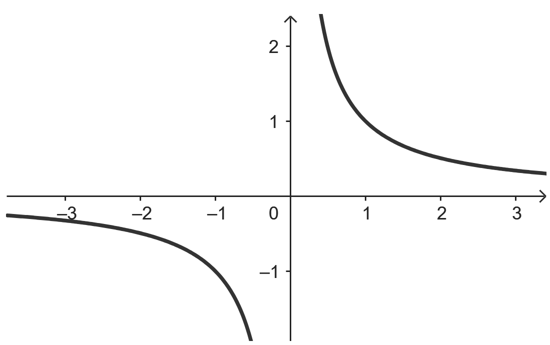
1) $\lim_{x \to 2} f(x) = 2$



2) $\lim_{x \to 2} f(x) = 2$

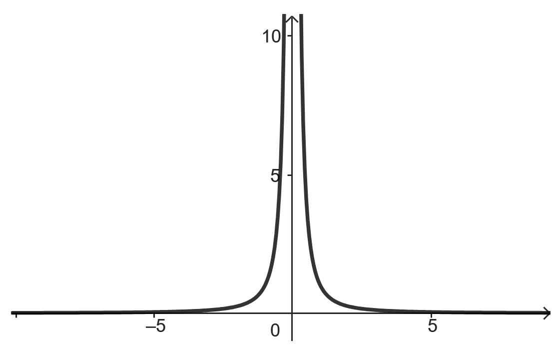


3) $\lim_{x \to 2} f(x) = 2$



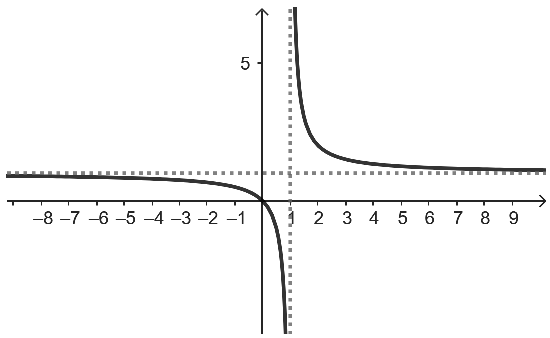
4) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

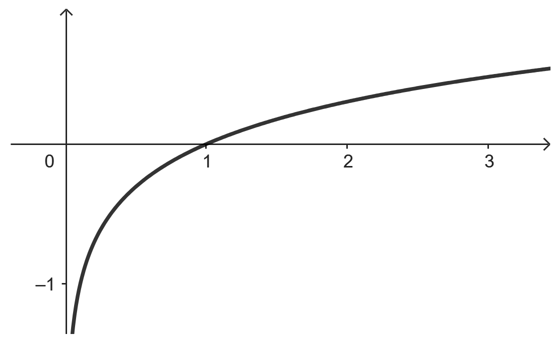
5) $\lim_{x \to 0^+} \frac{1}{x} = \infty$



6) $\lim_{x \to 0^-} \frac{1}{x^2} = \infty$

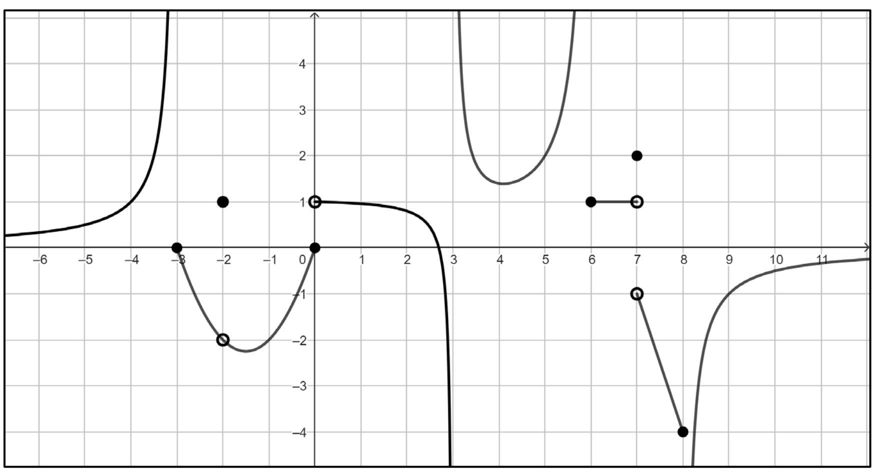
7) $\lim_{x \to 0^+} \frac{1}{x^2} = \infty$





8) $\lim\limits_{x\to1^-}\dfrac{x}{x-1}=-\infty$	**9)** $\lim\limits_{x\to1^+}\dfrac{x}{x-1}=\infty$	**10)** $\lim\limits_{x\to\infty}\log(x)=0$	**11)** $\lim\limits_{x\to0^+}\log(x)=-\infty$

12) Given the graph of $f(x)$ above, find the limit.

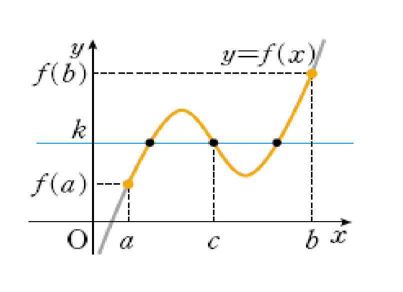
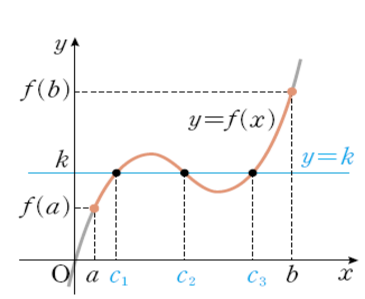
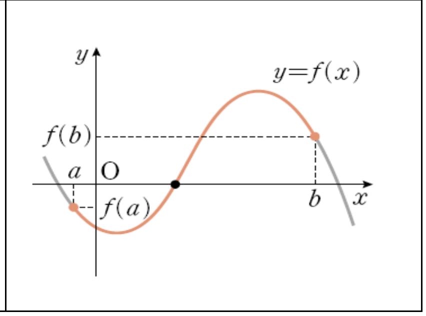


$\lim\limits_{x\to-\infty}f(x)=0$	$\lim\limits_{x\to\infty}f(x)=0$	$\lim\limits_{x\to-3^+}f(x)=0$	$\lim\limits_{x\to-3^-}f(x)=\infty$
$\lim\limits_{x\to-2^-}f(x)=-2$	$\lim\limits_{x\to-2^+}f(x)=-2$	$\lim\limits_{x\to0^+}f(x)=1$	$\lim\limits_{x\to0^-}f(x)=0$
$\lim\limits_{x\to3^+}f(x)=\infty$	$\lim\limits_{x\to3^-}f(x)=-\infty$	$\lim\limits_{x\to6^+}f(x)=1$	$\lim\limits_{x\to6^-}f(x)=-\infty$
$\lim\limits_{x\to7^+}f(x)=-1$	$\lim\limits_{x\to7^-}f(x)=1$	$\lim\limits_{x\to8^+}f(x)=-\infty$	$\lim\limits_{x\to8^-}f(x)=-4$
$f(-2)=1$	$f(7)=2$		

0-4. Basic Theorems in Calculus (Preview)

Theorem: Intermediate Value Theorem, IVT

If **f is continuous** on a closed interval [a, b] and k is any number between f(a) and f(b), then there exists at least one number c in (a, b) such that f(c) = k.

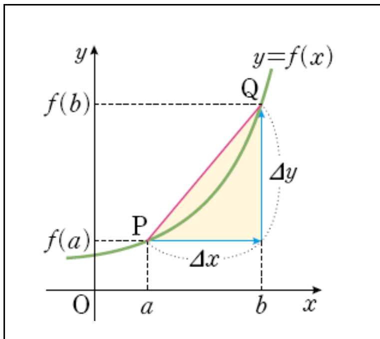
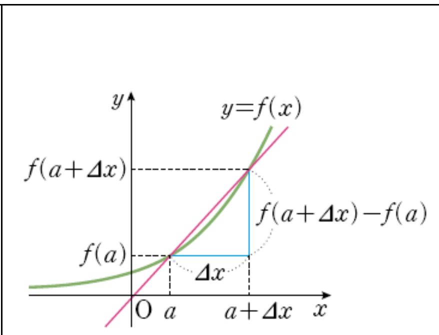
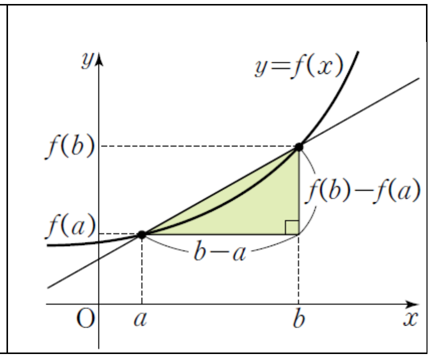
		If f(b)>0 and f(a)<0, then there exists at least one solution.
		

Definition of the Average Rate of Change

The average rate of change of y (slope m) with respect to x over the interval $[a,b]$ is given by:

$$m=\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}=\frac{f(a+\Delta x)-f(a)}{\Delta x}=\frac{f(a+h)-f(a)}{h}$$

where $h=\Delta x=b-a$.

Definition of the Instant Rate of Change

The graph demonstrates the concept of secant lines approaching the tangent line at a specific point on a curve as the interval between the points on the secant line (Δx) approaches zero. This visual representation helps in understanding the definition of the derivative, which is the slope of the tangent line at a given point on the function.

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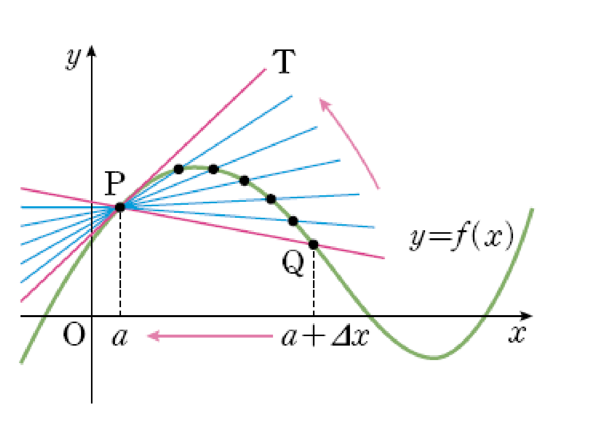
<tr>
<td>
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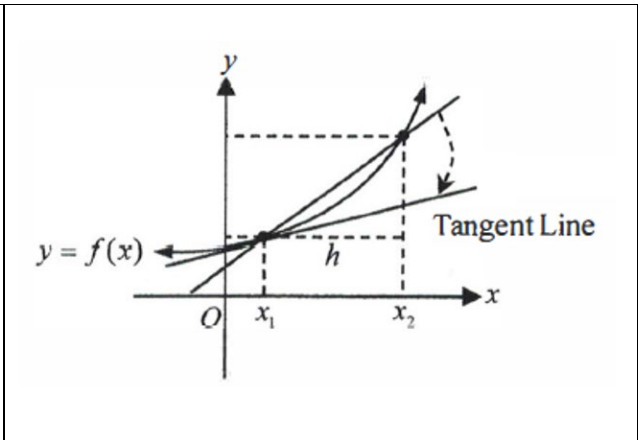
<td>
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<tr>
<td>$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$</td>
<td>$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$</td>
</tr>
</table>

- **Secant Lines**: The lines passing through points P and Q are secant lines. These lines intersect the curve at two points and approximate the slope of the function between those points.

  - As Δx decreases (meaning Q moves closer to P), the secant lines approach the slope of the tangent line at P.

- **Tangent Line T**:

  - Definition: The tangent line at point P is the line that just touches the curve at P without crossing it. This line represents the instantaneous rate of change of the function at $x = a$.
  - Slope of the Tangent Line: The slope of the tangent line at P is the limit of the slopes of the secant lines as Δx approaches zero.

- The slope of the tangent line at $x = a$ is given by the derivative: (replace Δx to h)

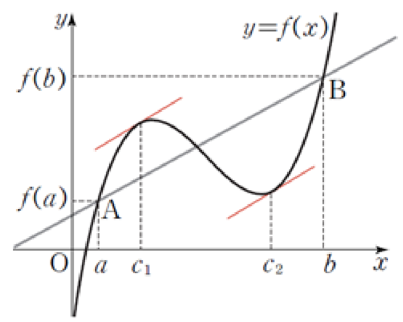
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

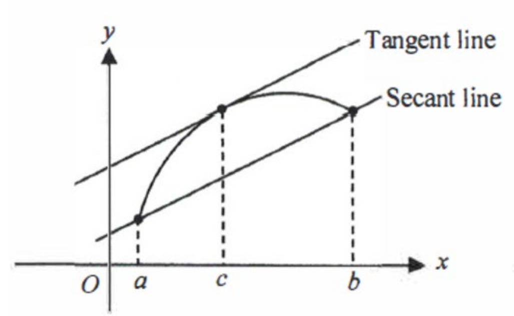
Theorem: The Mean-Value Theorem (MVT)

If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c between a and b such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

The slope of the secant line is equal to the slope of the tangent line.





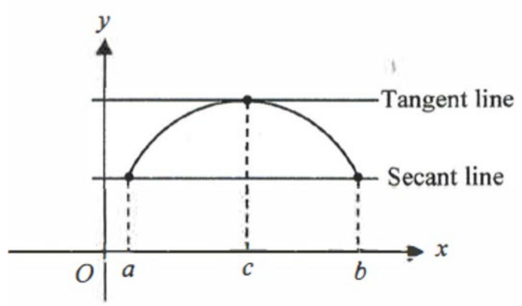
The Mean-Value Theorem guarantees that there is at least one point c in the interval (a, b) where the tangent line has the same slope as the secant line.

Theorem: Rolle's Theorem (Special case of the Mean Value Theorem)

Let f be continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

If $f(b)=f(a)$, then there exists at least one number c between a and b such that

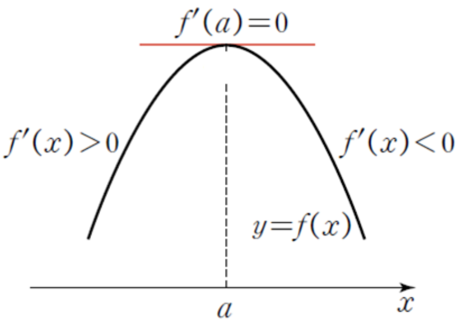
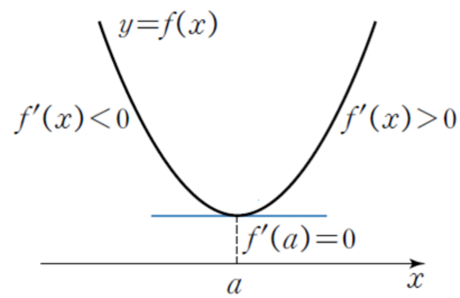
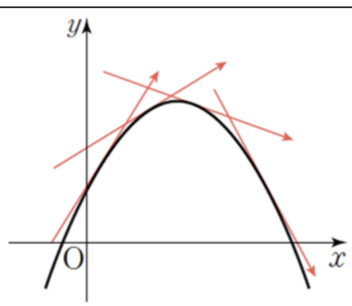
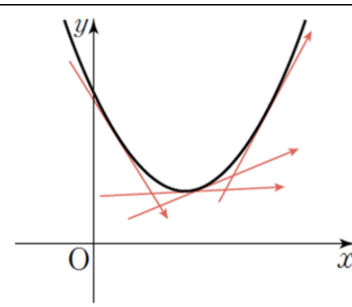
$$\frac{f(b)-f(a)}{b-a}=f'(c)=0.$$



Rolle's Theorem will guarantee the existence of an extreme value (relative maximum or relative minimum) in the interval.

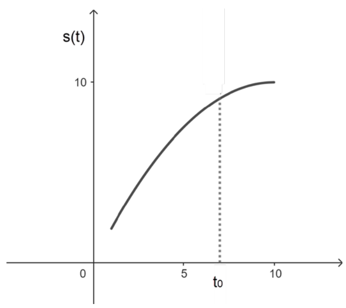
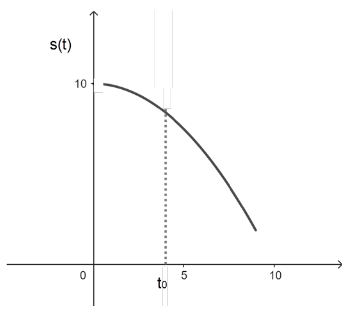
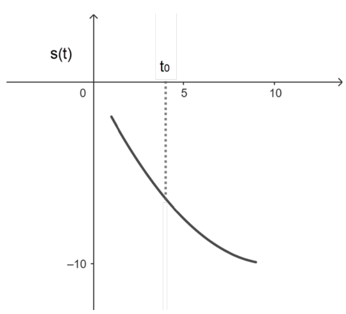
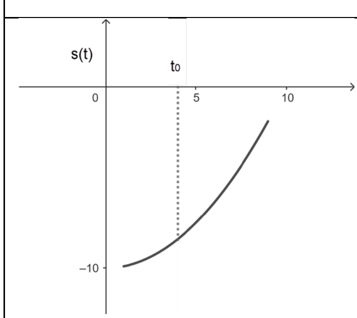
Maxima and Minima

- Maxima: At a local maximum, the derivative changes from positive to negative.
- Minima: At a local minimum, the derivative changes from negative to positive.
- Tangent lines at the critical points where $f'(a)=0$ confirm the behavior of the slopes, showing a peak for maxima and a valley for minima.

Maxima (relative maximum at x=a)	**Minima (relative minimum at x=a)**
 	
- Function $y=f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$. - Left of a: $f'(x)>0$, the function is increasing. - Right of a: $f'(x)<0$, the function is decreasing. - This indicates a local maximum at $x=a$.	- Function $y=f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$. - Left of a: $f'(x)<0$, the function is decreasing. - Right of a: $f'(x)>0$, the function is increasing. - This indicates a local minimum at $x=a$.
 	
- Shows the tangent lines with positive slopes approaching $x=a$ from the left and negative slopes after $x=a$, confirming the maximum. - Concave Downward	- Shows the tangent lines with negative slopes approaching $x=a$ from the left and positive slopes after $x=a$, confirming the minimum. - Concave Upward

0-5. Behavior of the Particle about Position vs. Time Curve (Preview)

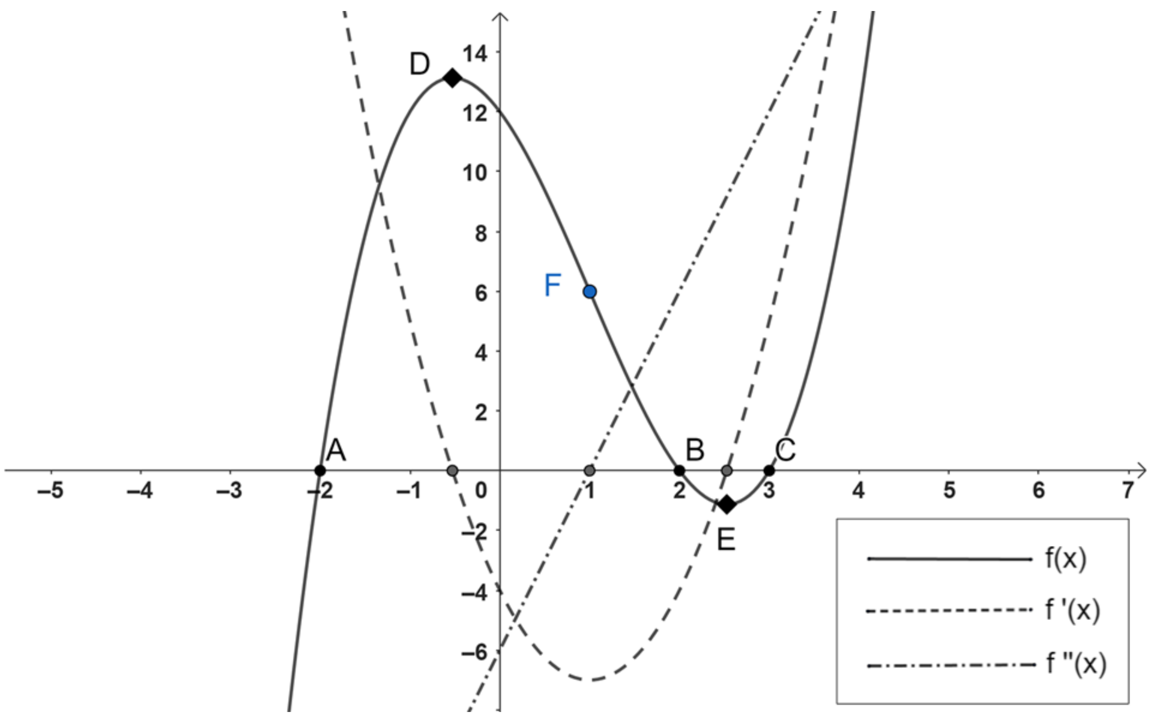
Observe behavior of the particle about the position versus time curve.

	At $t = t_0$ - Curve has positive slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is decreasing. - Particle is slowing down. - $v(t_0) > 0$ and $a(t_0) < 0$
	2. At $t = t_0$ - Curve has negative slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the negative direction. - Velocity is decreasing. - Particle is speeding up. - $v(t_0) < 0$ and $a(t_0) < 0$
	3. At $t = t_0$ - Curve has negative slope. - Curve is concave up. - $s(t_0) < 0$ - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the negative side of the origin. - Particle is moving in the negative direction. - Velocity is increasing (**slope** is increasing by t is progressing). - Particle is slowing down. - $v(t_0) < 0$ and $a(t_0) > 0$
	4. At $t = t_0$ - Curve has positive slope. - Curve is concave up. - $s(t_0) < 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is increasing. - Particle is speeding up. - $v(t_0) > 0$ and $a(t_0) > 0$

0-6. Derivative Test (Preview)

Concept Expansion from Pre-Calculus:

1) Find local maximum (D) and local minimum (E) values and the inflection point (F) for $f(x) = x^3 - 3x^2 - 4x + 12$ without using Calculus Concept?



Solution

- We can easily find roots (A, B & C) by factorization $\Rightarrow roots: x = -2, 2, 3$
- However, it is not easy to find x values for D (local max), E(local min) and F(Inflection point). Before calculus, to solve this problem, we may need to use the approximation method.
- Once we learn about derivatives, then we can find these points easily.

2) Sketch the polynomial graph of $f(x) = x^3 - 3x^2 - 24x + 32$ **by using** $f'(x)$**,** $f''(x)$

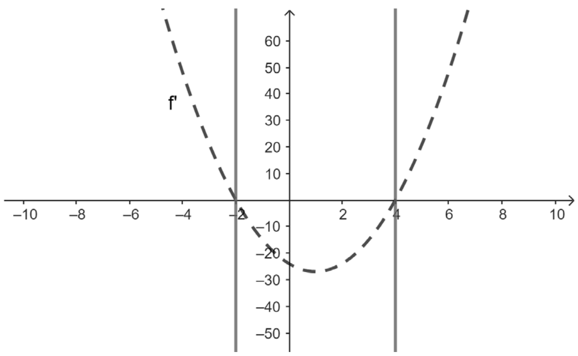
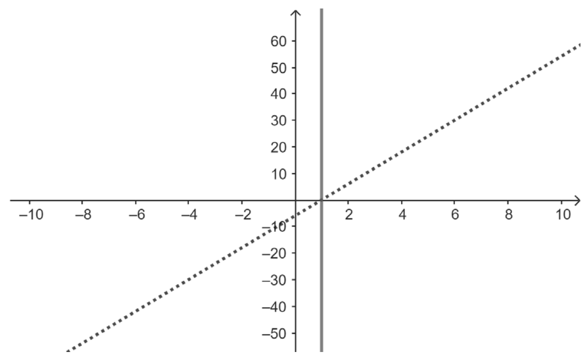
Solution Steps:

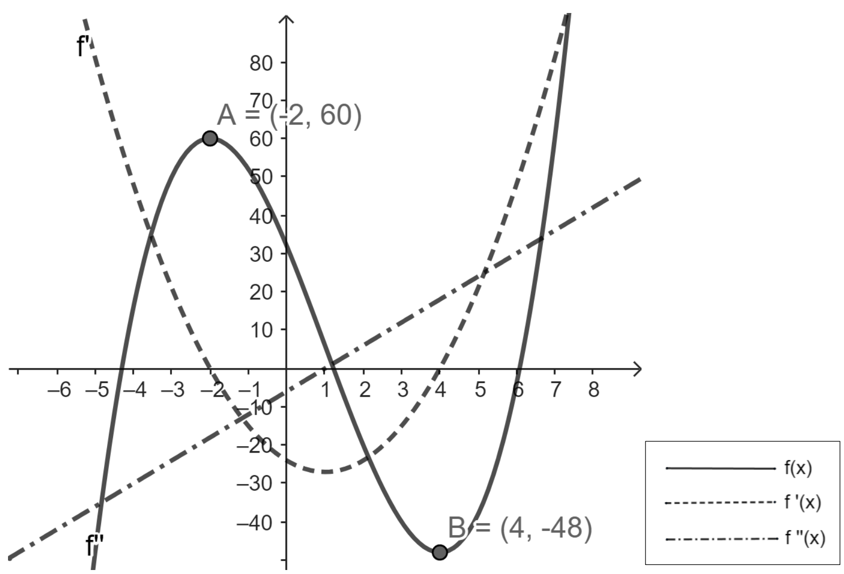
1. Find $f(x)$, $f'(x)$, $f''(x)$
   - $f(x) = x^3 - 3x^2 - 24x + 32$
   - $f'(x) = 3x^2 - 6x - 24$
   - $f''(x) = 6x - 6$
2. Find the first derivative ($f'(x)$) equal to zero to find **critical points** and its functional values if exist
   - $3x^2 - 6x - 24 = 0 \Rightarrow 3(x-4)(x+2) = 0 \Rightarrow x = 4, x = -2$
   - $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32 = 60$ (maxima)
   - $f(4) = (4)^3 - 3(4)^2 - 24(4) + 32 = -48$ (minima)
3. Set the second derivative **(** $f''(x)$ **)** equal to zero to find **inflection points** and its functional values if exist
   - $6x - 6 = 0 \Rightarrow 6(x-1) = 0 \Rightarrow x = 1$
   - $f(1) = (1)^3 - 3(1)^2 - 24(1) + 32 = 6$ (inflection point)
4. Determine the y-intercept
   - $f(0) = 0^3 - 3(0)^2 - 24(0) + 32 = 32$

5. Determine the **concavity and relative extrema** using the first and second derivatives
Number of critical points (including inflection points): 3, so need 4 sections on the graph

$f(x) = x^3 - 3x^2 - 24x + 32$							
$f'(x) = 3x^2 - 6x - 24$	**+**	Local Max $x = -2$	**-**	**-**	**-**	Local Min $x = 4$	**+**
	Increase	Critical Point	Decrease			Critical Point	Increase
$f''(x) = 6x - 6$	**-**	**-**	**-**	1	**+**	**+**	**+**
	Concave Down			Inflection Point	Concave Up		

6. Sketch the graph:

$f'(x)$	$f''(x)$
	



1. Plot the relative maximum at A $(-2,60)$.
2. Plot the relative minimum at B $(4,-48)$.
3. Plot the inflection point at C $(1,6)$.
4. Plot the y-intercept at D $(0,32)$.
5. Draw the curve concave down from $(-\infty,-2)$, then continue concave down through $(-2,60)$ to $(1,6)$.
6. Switch to concave up from $(1,6)$ to $(4,-48)$ and continue concave up to (∞,∞).

3) Sketch the rational function graph of $f(x)=\dfrac{x^2-4x+3}{x}$ by using $f'(x)$, $f''(x)$

To sketch the rational function $f(x)=\dfrac{x^2-4x+3}{x}$ using its first and second derivatives, follow these steps:

1. Simplify the Function: $f(x)=\dfrac{x^2-4x+3}{x}=x-4+\dfrac{3}{x}$

2. Find Asymptotes

- Vertical Asymptote: Occurs where the denominator is zero: $x=0$
- Slanted (oblique) Asymptote: $y=x-4$

3. Find Intercepts

- x-intercepts: Set $f(x)=0$: $x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$, So, $x=1$ and $x=3$.
- y-intercept: Set $x=0$: The function is undefined at $x=0$, so there is no y-intercept.

4. Find Critical Points (First Derivative)

- Find the first derivative $f'(x)$: $f'(x)=\dfrac{d}{dx}\left(x-4+\dfrac{3}{x}\right)=1-\dfrac{3}{x^2}$
- Set $f'(x)=0$: $1-\dfrac{3}{x^2}=0$ $x=\pm\sqrt{3}$

5. Find Points of Inflection (Second Derivative)

- Find the second derivative $f''(x)$: $f''(x)=\dfrac{d}{dx}\left(1-\dfrac{3}{x^2}\right)=\dfrac{6}{x^3}$
- Set $f''(x)=0$: $\dfrac{6}{x^3}=0$
- There are no real solutions. So, there are no points of inflection.

6. Analyze Intervals of Increase/Decrease

- For $x > 0$: $f'(x) = 1 - \frac{3}{x^2}$:
- If $x > \sqrt{3}$, $f'(x) > 0$ (positive).
- If $0 < x < \sqrt{3}$, $f'(x) < 0$ (negative).
- For $x < 0$: $f'(x) = 1 - \frac{3}{x^2}$:
- Always $f'(x) < 0$ (negative).

So, $x = -\sqrt{3}$ and $x = \sqrt{3}$ are **critical points:**

- Increasing on $(\sqrt{3}, \infty)$
- Decreasing on $(0, \sqrt{3})$ and $(-\infty, 0)$

7. Sketch the Graph

- **Asymptotes:** Vertical asymptote at $x = 0$, Horizontal asymptote at $y = 1$
- **Intercepts:** x-intercepts at $x = 1$ and $x = 3$
- Critical Points:
  - Local Minimum at $x = \sqrt{3}$: $y = -0.5$
  - Local Maximum at $x = -\sqrt{3}$: $y = -7.5$

Alternate simple way by graphing $f'(x)$ & $f''(x)$

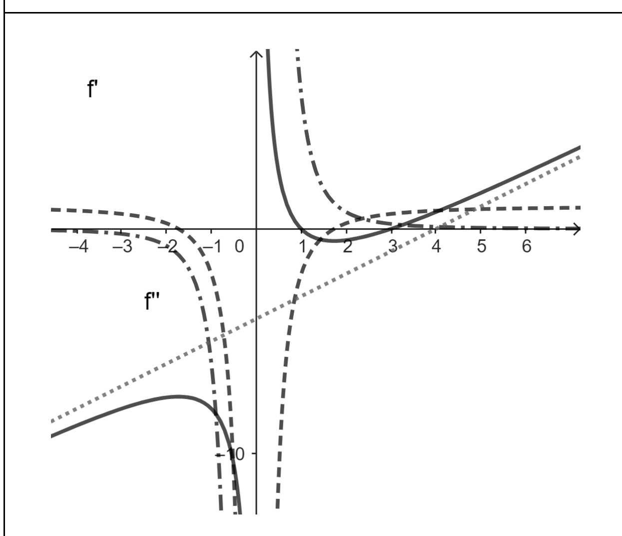
$$f(x)=\frac{x^2-3x-4}{x}$$

Find: (Refer below table)

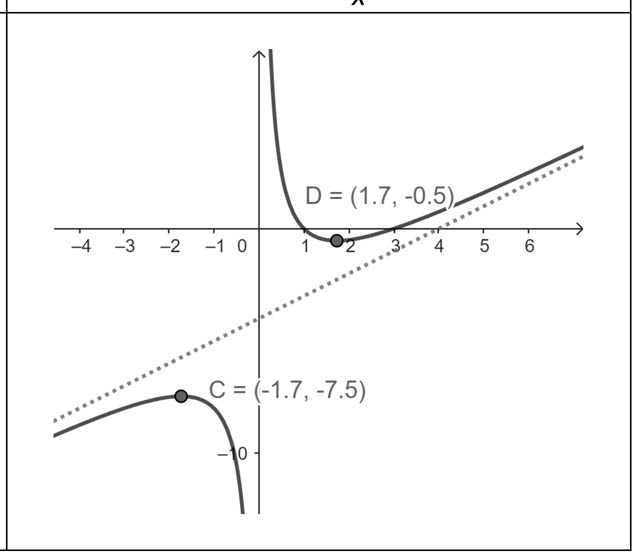
- Find $f'(x)$, then find all critical points **candidates** by factorizing if exists
- Find $f''(x)$, then find all inflection points **candidates** by factorizing if exists
- Find signs (positivity or negativity) before and after (all candidates) critical/inflection points
- If $f'(x)\geq 0$, the $f(x)$ is increasing on the ranges
- If $f'(x)\leq 0$, the $f(x)$ is decreasing on the ranges
- If $f''(x)\geq 0$, the $f(x)$ is concave-up on the ranges
- If $f''(x)\leq 0$, the $f(x)$ is concave-down the ranges
- Find all critical/inflection points from candidates, then find $f(x)$ values (y-values)
- If local min/min y-values from critical points.

$f'(x)=1-\frac{3}{x^2}$	+	Local Max $x=\sqrt{3}$	-	0 (undefined)	-	Local Min $x=-\sqrt{3}$	+
	Increase	Critical Point	decrease	Not critical point	decrease	Critical Point	Increase
$f''(x)=\frac{6}{x^3}$	-	-	-	0 (undefined)	+	+	+
	Concave Down			Inflection Point	Concave Down		

$f'(x)=1-\frac{3}{x^2}$, $f''(x)=\frac{6}{x^3}$



$f(x)=\frac{x^2-3x-4}{x}$



0-7. Derivative and antiderivative Formula (Preview)

1) Derivative and Integral Rules

	Derivative	Integral (Antiderivative)
1	$\frac{d}{dx}n=0$	$\int 0dx=C$
2	$\frac{d}{dx}x=1$	$\int 1dx=x+C$
3	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int\left[x^n\right]dx=\frac{x^{n+1}}{n+1}+C$
4	$\frac{d}{dx}\left[e^x\right]=e^x$	$\int\left[e^x\right]dx=e^x+C$
5	$\frac{d}{dx}[\ln x]=\frac{1}{x}$	$\int\left[\frac{1}{x}\right]dx=\ln x+C$
6	$\frac{d}{dx}\left[n^x\right]=n^x\ln n$	$\int\left[n^x\right]dx=\frac{n^x}{\ln n}+C$
7	$\frac{d}{dx}[\sin x]=\cos x$	$\int[\cos x]dx=\sin x+C$
8	$\frac{d}{dx}[\cos x]=-\sin x$	$\int[\sin x]dx=-\cos x+C$
9	$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int\left[\sec^2 x\right]dx=\tan x+C$
10	$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int\left[\csc^2 x\right]dx=-\cot x+C$
11	$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int[\tan x\sec x]dx=\sec x+C$
12	$\frac{d}{dx}[\csc x]=-\csc x\cot x$	$\int[\cot x\csc x]dx=-\csc x+C$
13	$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C$
14	$\frac{d}{dx}[\arccos x]=-\frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}dx=\arccos x+C$
15	$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$	$\int\frac{1}{1+x^2}dx=\arctan x+C$

16	$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \operatorname{arccot} x + C$
18	$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$
19	$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arccsc} x + C$

2) General Differentiation Rules

Let **c** be a real number, **n** be a rational number, **u** and **v** be differentiable functions of **x**, let **f** be a differentiable function of **u**, and let **a** be a positive real number ($a \neq 1$).

	Differentiation Rules	
1	Constant Rule	$\frac{d}{dx}[c] = 0$
2	Constant Multiple Rule	$\frac{d}{dx}[cu] = cu'$
3	Product Rule	$\frac{d}{dx}[uv] = uv' + vu'$
4	Chain Rule	$\frac{d}{dx}[f(u)] = f'(u)u'$
5	(Simple) Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$
6	Sum or Difference Rule	$\frac{d}{dx}[u \pm v] = u' \pm v'$
7	Quotient Rule	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
8	General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
9	Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

		$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$
10	Derivatives of Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$ $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$ $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
11	Derivatives of Inverse Trigonometric Functions	$\frac{d}{dx}\left[\arcsin x\right] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\arccos x\right] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\arctan x\right] = \frac{1}{1+x^2}$ $\frac{d}{dx}\left[\text{arccot}\, x\right] = -\frac{1}{1+x^2}$ $\frac{d}{dx}\left[\text{arcsec}\, x\right] = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}\left[\text{arccsc}\, x\right] = -\frac{1}{x\sqrt{x^2-1}}$
12	Derivatives of Inverse Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$ $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\|u\|\sqrt{u^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{\|u\|\sqrt{u^2-1}}$

13	Derivatives of Basic Hyperbolic Trigonometric Functions $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$ $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$	$\frac{d}{dx}[\sinh(x)]=\cosh(x)$ $\frac{d}{dx}[\cosh(x)]=\sinh(x)$ $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$ $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$ $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$ $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$
14	Derivatives of Inverse Hyperbolic Trigonometric Functions	$\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$ $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$ $\frac{d}{dx}[\text{sech}^{-1}(x)]=\frac{-1}{x\sqrt{1-x^2}}$ $\frac{d}{dx}[\text{csch}^{-1}(x)]=\frac{-1}{\|x\|\sqrt{1+x^2}}$ $\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$
15	Derivatives of Exponential and Logarithmic Functions	$\frac{d}{dx}[e^x]=e^x$ $\frac{d}{dx}[\ln x]=\frac{1}{x}$ $\frac{d}{dx}[a^x]=(\ln a)a^x$ $\frac{d}{dx}[\log_a x]=\frac{1}{(\ln a)x}$
16	Basic Differentiation Rules for Elementary Functions (**u & v** be differentiable functions of **x)**	$\frac{d}{dx}[u^n]=nu^{n-1}u'$ $\frac{d}{dx}[\|u\|]=\frac{u}{\|u\|}u', \quad u\neq 0$ $\frac{d}{dx}[\ln u]=\frac{u'}{u}$ $\frac{d}{dx}[e^u]=e^u u'$

		$\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$ $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

3) Hyperbolic functions are analogs of the circular trigonometric functions, but for a hyperbola. They are extensively used in various areas of mathematics, including algebra, calculus, and complex analysis. Here are the basic hyperbolic functions along with their definitions:

1	Hyperbolic Sine ($\sinh x$)	$\sinh x = \frac{e^x - e^{-x}}{2}$
2	Hyperbolic Cosine ($\cosh x$)	$\cosh x = \frac{e^x + e^{-x}}{2}$
3	Hyperbolic Tangent ($\tanh x$)	$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4	Hyperbolic Cosecant ($\text{csch } x$)	$\text{csch } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
5	Hyperbolic Secant ($\text{sech } x$)	$\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
6	Hyperbolic Cotangent ($\coth x$)	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

4) List of antiderivative Formulas: covering a wider range of functions. These include basic functions, exponential and logarithmic functions, trigonometric functions, and some of their inverses

	Functions	Antiderivative Formulas:
1	Constant Function	$\int a\,dx = ax + C$
2	Power Function	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3	Exponential Function	$\int e^x\,dx = e^x + C$
4	General Exponential Function	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$
5	Natural Logarithm	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$

6	Sine Functions	$\int \sin(x)dx = -\cos(x) + C$
7	Cosine Functions	$\int \cos(x)dx = \sin(x) + C$
8	Tangent Functions	$\int \tan(x)dx = -\ln\|\cos(x)\| + C$
9	Cotangent (cot) Functions	$\int \cot(x)dx = \ln\|\sin(x)\| + C$
10	Secant (sec) Functions	$\int \sec(x)dx = \ln\|\sec(x) + \tan(x)\| + C$
11	Cosecant (csc) Functions	$\int \csc(x)dx = -\ln\|\csc(x) + \cot(x)\| + C$
12	Inverse Sine (arcsin) Functions	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$
13	Inverse Tangent (arctan) Functions	$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$
14	sinh (Hyperbolic Sine) Functions	$\int \sinh(x)dx = \cosh(x) + C$
15	cosh (Hyperbolic Cosine) Functions	$\int \cosh(x)dx = \sinh(x) + C$
16	Integral of sec^2	$\int \sec^2(x)dx = \tan(x) + C$
17	Integral of csc^2	$\int \csc^2(x)dx = -\cot(x) + C$
18	Integral of $\sec(x)\tan(x)$	$\int \sec(x)\tan(x)dx = \sec(x) + C$
19	Integral of $\csc(x)\cot(x)$	$\int \csc(x)\cot(x)dx = -\csc(x) + C$

0-8. Find Derivatives (Preview)

Example: Solve all

1) Find the derivative of the function $f(x) = 7$. (Constant Rule)

- Using the constant rule, which states $\frac{d}{dx}[c] = 0$
- $f'(x) = \frac{d}{dx}[7] = 0$

2) Find the derivative of the function $f(x) = 5x^3$. (Constant Multiple Rule)

- Using the constant multiple rule, which states $\frac{d}{dx}[cu] = cu'$
- $f'(x) = \frac{d}{dx}[5x^3] = 5 \cdot \frac{d}{dx}[x^3] = 5 \cdot 3x^2 = 15x^2$

3) Find the derivative of the function $f(x) = x^2 \sin(x)$. (Product Rule)

- Using the product rule, which states $\frac{d}{dx}[uv] = uv' + vu'$
- $u = x^2$, $v = \sin(x)$
- $u' = 2x$, $v' = \cos(x)$
- $f'(x) = (x^2)' \sin(x) + x^2 (\sin(x))' = 2x \sin(x) + x^2 \cos(x)$

4) Find the derivative of the function $f(x)=\sin(3x)$. (Chain Rule)

- Using the chain rule, which states $\frac{d}{dx}[f(u)]=f'(u)u'$
- $f(u)=\sin(u)$, $u=3x$
- $f'(u)=\cos(u)$, $u'=3$
- $f'(x)=\cos(3x)\cdot 3=3\cos(3x)$

5) Find the derivative of the function $f(x)=x^5$. ((Simple) Power Rule)

- Using the power rule, which states $\frac{d}{dx}[x^n]=nx^{n-1}$
- $f'(x)=\frac{d}{dx}[x^5]=5x^4$

6) Find the derivative of the function $f(x)=x^3-4x+7$. (Sum or Difference Rule)

- Using the sum or difference rule, which states $\frac{d}{dx}[u\pm v]=u'\pm v'$
- $f'(x)=\frac{d}{dx}[x^3]-\frac{d}{dx}[-4x]+\frac{d}{dx}[7]=3x^2-4+0=3x^2-4$

7) Find the derivative of the function $f(x) = \frac{x^2}{\sin(x)}$**. (Quotient Rule)**

- Using the quotient rule, which states $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
- $u = x^2$, $v = \sin(x)$
- $u' = 2x$, $v' = \cos(x)$
- $f'(x) = \frac{\sin(x) \cdot 2x - x^2 \cdot \cos(x)}{\sin^2(x)} = \frac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)}$

8) Find the derivative of the function $f(x) = (3x^2 + 2)^4$**. (General Power Rule)**

- Using the general power rule, which states $\frac{d}{dx}[u^n] = nu^{n-1}u'$
- $u = 3x^2 + 2$, $u' = 6x$
- $n = 4$
- $f'(x) = 4(3x^2 + 2)^3 \cdot 6x = 24x(3x^2 + 2)^3$

9) Find the derivative of the function $f(x) = \tan(x)$**. (Derivatives of Trigonometric Functions)**

- Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(x)] = \sec^2(x)$
- $f'(x) = \frac{d}{dx}[\tan(x)] = \sec^2(x)$

10) Find the derivative of the function $f(x)=\sin(x)$. (Derivative of $\sin(x)$)

- Using the derivative rule for the sine function, which states $\frac{d}{dx}[\sin(x)]=\cos(x)$
- $f'(x)=\frac{d}{dx}[\sin(x)]=\cos(x)$

11) Find the derivative of the function $f(x)=\cos(x)$. (Derivative of $\cos(x)$)

- Using the derivative rule for the cosine function, which states $\frac{d}{dx}[\cos(x)]=-\sin(x)$
- $f'(x)=\frac{d}{dx}[\cos(x)]=-\sin(x)$

12) Find the derivative of the function $f(x)=\tan(2x)$. (Derivative of $\tan(x)$)

- Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(u)]=\sec^2(u)\cdot u'$
- $f'(x)=\frac{d}{dx}[\tan(2x)]=\sec^2(2x)\cdot 2$

13) Find the derivative of the function $f(x)=\cot(x)$. (Derivative of $\cot(x)$)

- Using the derivative rule for the cotangent function, which states $\frac{d}{dx}[\cot(x)]=-\csc^2(x)$
- $f'(x)=\frac{d}{dx}[\cot(x)]=-\csc^2(x)$

14) Find the derivative of the function $f(x)=\sec(x)$. (Derivative of $\sec(x)$)

- Using the derivative rule for the secant function, which states $\frac{d}{dx}[\sec(x)]=\sec(x)\tan(x)$
- $f'(x)=\frac{d}{dx}[\sec(x)]=\sec(x)\tan(x)$

15) Find the derivative of the function $f(x)=\csc(x)$. (Derivative of $\csc(x)$)

- Using the derivative rule for the cosecant function, which states $\frac{d}{dx}[\csc(x)]=-\csc(x)\cot(x)$
- $f'(x)=\frac{d}{dx}[\csc(x)]=-\csc(x)\cot(x)$

16) Find the derivative of the function $f(x)=\sin(3x^2+2x)$. (Derivative of $\sin(u)$ where u is a function of x)

- Using the chain rule and the derivative of sine, which states $\frac{d}{dx}[\sin(u)]=(\cos(u))u'$
- $u=3x^2+2x$, $u'=6x+2$
- $f'(x)=\cos(3x^2+2x)\cdot(6x+2)=(6x+2)\cos(3x^2+2x)$

17) Find the derivative of the function $f(x)=\cos(x^3-x)$ **. (Derivative of** $\cos(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of cosine, which states $\frac{d}{dx}[\cos(u)]=-(\sin(u))u'$
- $u=x^3-x$, $u'=3x^2-1$
- $f'(x)=-\sin(x^3-x)\cdot(3x^2-1)=-(3x^2-1)\sin(x^3-x)$

18) Find the derivative of the function $f(x)=\tan(2x^2-3x)$ **. (Derivative of** $\tan(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of tangent, which states $\frac{d}{dx}[\tan(u)]=(\sec^2(u))u'$
- $u=2x^2-3x$, $u'=4x-3$
- $f'(x)=\sec^2(2x^2-3x)\cdot(4x-3)=(4x-3)\sec^2(2x^2-3x)$

19) Find the derivative of the function $f(x)=\cot(4x^3+x^2)$ **. (Derivative of** $\cot(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of cotangent, which states $\frac{d}{dx}[\cot(u)]=-(\csc^2(u))u'$
- $u=4x^3+x^2$, $u'=12x^2+2x$
- $f'(x)=-\csc^2(4x^3+x^2)\cdot(12x^2+2x)=-(12x^2+2x)\csc^2(4x^3+x^2)$

20) Find the derivative of the function $f(x) = \sec(3x^2 + x)$. (Derivative of $\sec(u)$ where u is a function of x)

- Using the chain rule and the derivative of secant, which states $\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))u'$
- $u = 3x^2 + x$, $u' = 6x + 1$
- $f'(x) = \sec(3x^2 + x)\tan(3x^2 + x)\cdot(6x+1) = (6x+1)\sec(3x^2 + x)\tan(3x^2 + x)$

21) Find the derivative of the function $f(x) = \csc(x^2 + 2x)$. (Derivative of $\csc(u)$ where u is a function of x)

- Using the chain rule and the derivative of cosecant, which states $\frac{d}{dx}[\csc(u)] = -(\csc(u)\cot(u))u'$
- $u = x^2 + 2x$, $u' = 2x + 2$
- $f'(x) = -\csc(x^2 + 2x)\cot(x^2 + 2x)\cdot(2x+2) = -(2x+2)\csc(x^2 + 2x)\cot(x^2 + 2x)$

22) Find the derivative of the function $f(x) = \sinh(x)$. (Derivative of $\sinh(x)$)

- Using the derivative rule for the hyperbolic sine function, which states $\frac{d}{dx}[\sinh(x)] = \cosh(x)$
- $f'(x) = \frac{d}{dx}[\sinh(x)] = \cosh(x)$

23) Find the derivative of the function $f(x)=\cosh(x)$. (Derivative of $\cosh(x)$)

- Using the derivative rule for the hyperbolic cosine function, which states $\frac{d}{dx}[\cosh(x)]=\sinh(x)$
- $f'(x)=\frac{d}{dx}[\cosh(x)]=\sinh(x)$

24) Find the derivative of the function $f(x)=\tanh(x)$. (Derivative of $\tanh(x)$)

- Using the derivative rule for the hyperbolic tangent function, which states $\frac{d}{dx}[\tanh(x)]=\operatorname{sech}^2(x)$
- $f'(x)=\frac{d}{dx}[\tanh(x)]=\operatorname{sech}^2(x)$

25) Find the derivative of the function $f(x)=\operatorname{sech}(x)$. (Derivative of $\operatorname{sech}(x)$)

- Using the derivative rule for the hyperbolic secant function, which states $\frac{d}{dx}[\operatorname{sech}(x)]=-\operatorname{sech}(x)\tanh(x)$
- $f'(x)=\frac{d}{dx}[\operatorname{sech}(x)]=-\operatorname{sech}(x)\tanh(x)$

26) Find the derivative of the function $f(x)=\operatorname{csch}(x)$. (Derivative of $\operatorname{csch}(x)$)

- Using the derivative rule for the hyperbolic cosecant function, which states $\frac{d}{dx}[\operatorname{csch}(x)]=-\operatorname{csch}(x)\coth(x)$
- $f'(x)=\frac{d}{dx}[\operatorname{csch}(x)]=-\operatorname{csch}(x)\coth(x)$

27) Find the derivative of the function $f(x) = \coth(x)$. (Derivative of $\coth(x)$)

- Using the derivative rule for the hyperbolic cotangent function, which states $\frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$
- $f'(x) = \frac{d}{dx}[\coth(x)] = -\operatorname{csch}^2(x)$

28) Find the derivative of the function $f(x) = \sinh^{-1}(x)$. (Derivative of $\sinh^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic sine function, which states $\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$
- $f'(x) = \frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$

29) Find the derivative of the function $f(x) = \cosh^{-1}(x)$. (Derivative of $\cosh^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic cosine function, which states $\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$
- $f'(x) = \frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$

30) Find the derivative of the function $f(x) = \tanh^{-1}(x)$. **(Derivative of** $\tanh^{-1}(x)$**)**

- Using the derivative rule for the inverse hyperbolic tangent function, which states $\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$
- $f'(x) = \frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$

31) Find the derivative of the function $f(x) = \text{sech}^{-1}(x)$. **(Derivative of** $\text{sech}^{-1}(x)$**)**

- Using the derivative rule for the inverse hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$
- $f'(x) = \frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$

32) Find the derivative of the function $f(x) = \text{csch}^{-1}(x)$. **(Derivative of** $\text{csch}^{-1}(x)$**)**

- Using the derivative rule for the inverse hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{|x|\sqrt{1+x^2}}$
- $f'(x) = \frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{|x|\sqrt{1+x^2}}$

33) Find the derivative of the function $f(x)=\coth^{-1}(x)$. (Derivative of $\coth^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic cotangent function, which states $\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$
- $f'(x)=\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$

34) Find the derivative of the function $f(x)=e^x$. (Derivative of e^x)

- Using the derivative rule for the exponential function, which states $\frac{d}{dx}[e^x]=e^x$
- $f'(x)=\frac{d}{dx}[e^x]=e^x$

35) Find the derivative of the function $f(x)=\ln(x)$. (Derivative of $\ln(x)$)

- Using the derivative rule for the natural logarithm function, which states $\frac{d}{dx}[\ln(x)]=\frac{1}{x}$
- $f'(x)=\frac{d}{dx}[\ln(x)]=\frac{1}{x}$

36) Find the derivative of the function $f(x)=2^x$. (Derivative of a^x)

- Using the derivative rule for the exponential function with base a, which states $\frac{d}{dx}[a^x]=(\ln a)a^x$
- $f'(x)=\frac{d}{dx}[2^x]=(\ln 2)2^x$

37) Find the derivative of the function $f(x)=\log_2(x)$. (Derivative of $\log_a(x)$)

- Using the derivative rule for the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(x)]=\frac{1}{(\ln a)x}$
- $f'(x)=\frac{d}{dx}[\log_2(x)]=\frac{1}{(\ln 2)x}$

38) Find the derivative of the function $f(x)=(3x^2+2)^5$. (Derivative of u^n)

- Using the general power rule, which states $\frac{d}{dx}[u^n]=nu^{n-1}u'$
- $u=3x^2+2$, $u'=6x$
- $f'(x)=5(3x^2+2)^4\cdot 6x=30x(3x^2+2)^4$

39) Find the derivative of the function $f(x)=|3x-4|$. (Derivative of |u|)

- Using the rule for the derivative of the absolute value function, which states $\frac{d}{dx}[|u|]=\frac{u}{|u|}u'$ where $u\neq 0$
- $u=3x-4$, $u'=3$
- $f'(x)=\frac{3x-4}{|3x-4|}\cdot 3=\frac{3(3x-4)}{|3x-4|}$

40) Find the derivative of the function $f(x)=\ln(2x^3+5)$. (Derivative of $\ln(u)$)

- Using the rule for the derivative of the natural logarithm function, which states $\frac{d}{dx}[\ln(u)]=\frac{u'}{u}$
- $u=2x^3+5$, $u'=6x^2$
- $f'(x)=\frac{6x^2}{2x^3+5}$

41) Find the derivative of the function $f(x)=e^{4x^2}$. (Derivative of e^u)

- Using the rule for the derivative of the exponential function, which states $\frac{d}{dx}[e^u]=e^u u'$
- $u=4x^2$, $u'=8x$
- $f'(x)=e^{4x^2}\cdot 8x=8xe^{4x^2}$

42) Find the derivative of the function $f(x)=\log_3(x^2+1)$. (Derivative of $\log_a(u)$)

- Using the rule for the derivative of the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(u)]=\frac{u'}{(\ln a)u}$
- $u=x^2+1$, $u'=2x$
- $f'(x)=\frac{2x}{(\ln 3)(x^2+1)}=\frac{2x}{(\ln 3)(x^2+1)}$

43) Find the derivative of the function $f(x)=5^{3x}$. (Derivative of a^u)

- Using the rule for the derivative of the exponential function with base a, which states $\frac{d}{dx}[a^u]=(\ln a)a^u u'$
- $u=3x$, $u'=3$
- $f'(x)=(\ln 5)5^{3x}\cdot 3=3(\ln 5)5^{3x}$

0-9. Find Antiderivatives (Preview)

Example: Solve all

1) Find the antiderivative of $\int 6dx$.

- Using the antiderivative formula for a constant function, $\int adx = ax + C$
- $\int 6dx = 6x + C$

2) Find the antiderivative of $\int x^3 dx$.

- Using the power rule for antiderivatives, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq -1$
- $\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$

3) Find the antiderivative of $\int e^x dx$.

- Using the antiderivative formula for the exponential function, $\int e^x dx = e^x + C$
- $\int e^x dx = e^x + C$

4) Find the antiderivative of $\int 3^x\,dx$.

- Using the antiderivative formula for the general exponential function, $\int a^x\,dx = \frac{a^x}{\ln(a)} + C$ where $a > 0, a \neq 1$
- $\int 3^x\,dx = \frac{3^x}{\ln(3)} + C$

5) Find the antiderivative of $\int \frac{1}{x}dx$.

- Using the antiderivative formula for the natural logarithm, $\int \frac{1}{x}dx = \ln|x| + C$
- $\int \frac{1}{x}dx = \ln|x| + C$

6) Find the antiderivative of $\int \sin(x)dx$.

- Using the antiderivative formula for the sine function, $\int \sin(x)dx = -\cos(x) + C$
- $\int \sin(x)dx = -\cos(x) + C$

7) Find the antiderivative of $\int \cos(x)dx$.

- Using the antiderivative formula for the cosine function, $\int \cos(x)dx = \sin(x) + C$
- $\int \cos(x)dx = \sin(x) + C$

8) Find the antiderivative of $\int \tan(x)dx$ **.**

- Using the antiderivative formula for the tangent function, $\int \tan(x)dx = -\ln|\cos(x)| + C$,
- $\int \tan(x)dx = -\ln|\cos(x)| + C$

9) Find the antiderivative of $\int \cot(x)dx$ **.**

- Using the antiderivative formula for the cotangent function, $\int \cot(x)dx = \ln|\sin(x)| + C$
- $\int \cot(x)dx = \ln|\sin(x)| + C$

10) Find the antiderivative of $\int \sec(x)dx$ **.**

- Using the antiderivative formula for the secant function, $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$

11) Find the antiderivative of $\int \csc(x)dx$ **.**

- Using the antiderivative formula for the cosecant function, $\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C$
- $\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C$

12) Find the antiderivative of $\int \frac{1}{\sqrt{1-x^2}}dx$ **.**

- Using the antiderivative formula for the inverse sine function, $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$

13) Find the antiderivative of $\int \frac{1}{1+x^2}dx$ **.**

- Using the antiderivative formula for the inverse tangent function, $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$
- $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$

14) Find the antiderivative of $\int \sinh(x)dx$ **.**

- Using the antiderivative formula for the hyperbolic sine function, $\int \sinh(x)dx = \cosh(x) + C$
- $\int \sinh(x)dx = \cosh(x) + C$

15) Find the antiderivative of $\int \cosh(x)dx$ **.**

- Using the antiderivative formula for the hyperbolic cosine function, $\int \cosh(x)dx = \sinh(x) + C$
- $\int \cosh(x)dx = \sinh(x) + C$

16) Find the antiderivative of $\int \sec^2(x)dx$.

- Using the antiderivative formula for the integral of $\sec^2(x)$, $\int \sec^2(x)dx = \tan(x) + C$
- $\int \sec^2(x)dx = \tan(x) + C$

17) Find the antiderivative of $\int \csc^2(x)dx$.

- Using the antiderivative formula for the integral of $\csc^2(x)$, $\int \csc^2(x)dx = -\cot(x) + C$
- $\int \csc^2(x)dx = -\cot(x) + C$

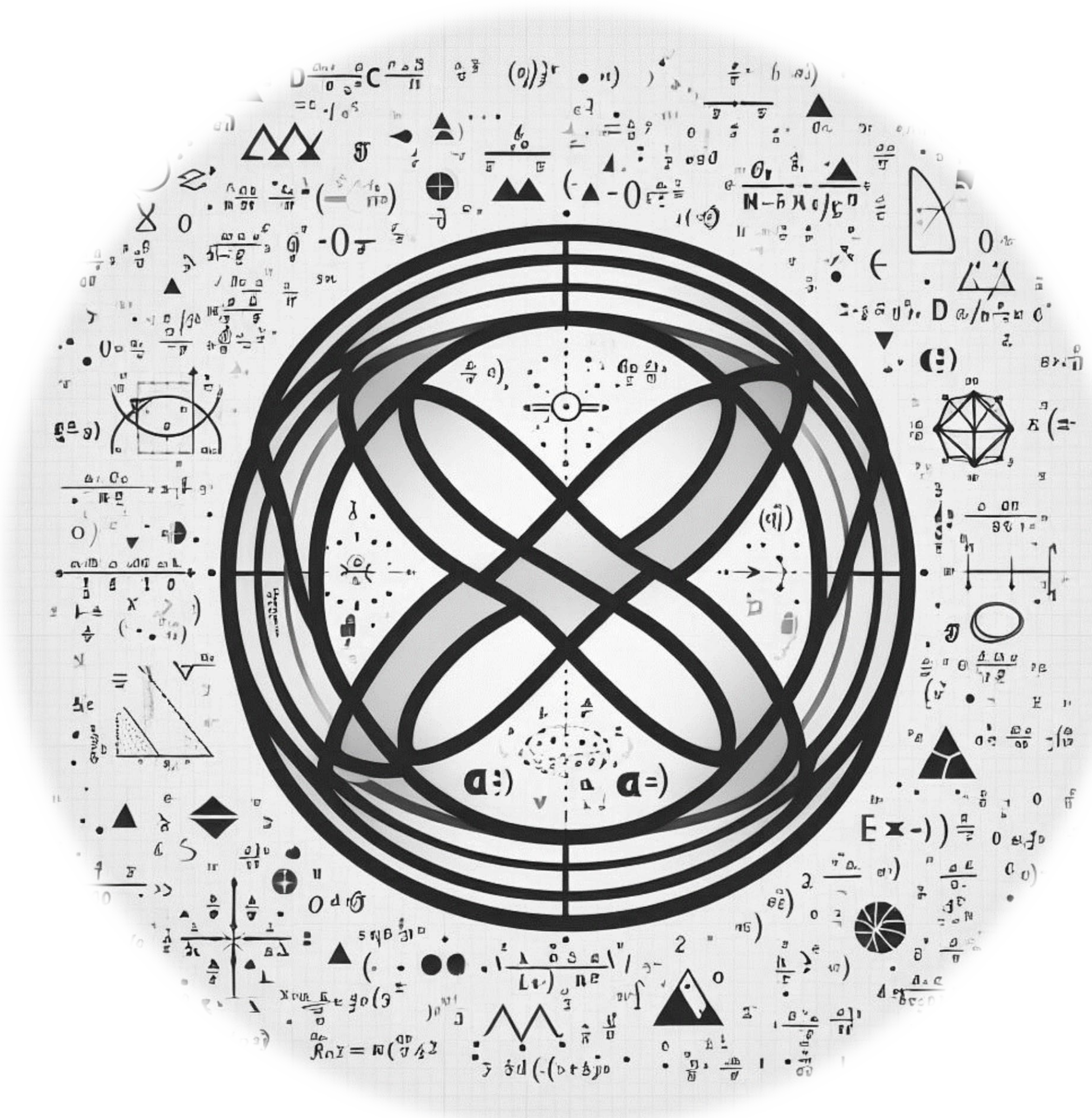
18) Find the antiderivative of $\int \sec(x)\tan(x)dx$.

- Using the antiderivative formula for the integral of $\sec(x)\tan(x)$, $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$

19) Find the antiderivative of $\int \csc(x)\cot(x)dx$.

- Using the antiderivative formula for the integral of $\csc(x)\cot(x)$, $\int \csc(x)\cot(x)dx = -\csc(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

Chapter 9. Differential Equations



9-1. Definitions and Basic Concepts of Differential Equations

Differential equations are mathematical equations that relate some function with its derivatives. They arise wherever a deterministic relationship involving some continuously varying quantities and their rates of change in space and/or time are known or postulated. They are pivotal in fields such as physics, engineering, economics, biology, and many others for modeling dynamic systems.

Types of Differential Equations:

1. Ordinary Differential Equations (**ODEs**): These involve functions of a single variable and their derivatives.

2. Partial Differential Equations (**PDEs**): These involve functions of multiple variables and their partial derivatives.

Order and Degree:

- **Order**: The order of a differential equation is the order of the highest derivative that appears in the equation. For example, an equation involving $\frac{d^3y}{dx^3}$ is a third-order equation.
- **Degree**: The degree of a differential equation is the power of the highest derivative in the equation, provided the equation is polynomial in derivatives.

Linearity:

- A differential equation is called linear if it can be expressed as a linear combination of the derivatives of y (including y itself), plus possibly a function of x . Otherwise, it is non-linear.

Initial Conditions and Boundary Conditions:

- **Initial Conditions**: Values specified for the function and its derivatives at a starting point. For ODEs, these are essential for uniquely determining a solution.
- **Boundary Conditions**: Constraints given for the values of the solution at the boundaries of the domain, typically used in solving PDEs.

Formulas:

1. Definition:

A linear first-order ordinary differential equation is an equation of the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

where y is the dependent variable, x is the independent variable, $p(x)$ is a function of x , and $q(x)$ is another function of x .

2. Solution Method:

The general solution to a linear first-order ODE can be found using an integrating factor. The integrating factor, $\mu(x)$, is given by:

$$\mu(x) = e^{\int p(x)dx}$$

3. Steps to Solve:

1. Identify $p(x)$ and $q(x)$: These are the functions of x in the standard form of the ODE.
2. Compute the integrating factor $\mu(x)$: $\mu(x) = e^{\int p(x)dx}$
3. Multiply through by the integrating factor: $\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$
4. Recognize that the left-hand side is the derivative of $\mu(x)y$: $\frac{d}{dx}[\mu(x)y] = \mu(x)q(x)$
5. Integrate both sides with respect to x : $\int \frac{d}{dx}[\mu(x)y]dx = \int \mu(x)q(x)dx$
6. Solve for y : $\mu(x)y = \int \mu(x)q(x)dx + C$

   $y = \frac{1}{\mu(x)}\left(\int \mu(x)q(x)dx + C\right)$ where C is the constant of integration.

Examples:

1) Solve the differential equation: $\frac{dy}{dx} = 4x$

- To solve the differential equation, we integrate both sides with respect to x:
  $\int \frac{dy}{dx} dx = \int 4x dx \Rightarrow \int dy = \int 4x dx \Rightarrow y = 4\int x dx$
- $y = 4\left(\frac{x^2}{2}\right) + C = 2x^2 + C$
- where C is the constant of integration.

2) Solve the initial value problem (IVP): $\frac{dy}{dx} = e^x$ **with the initial condition** $y(0) = 2$ **.**

- First, find the general solution by integrating both sides with respect to x:
  $\int \frac{dy}{dx} dx = \int e^x dx \Rightarrow \int dy = \int e^x dx \Rightarrow y = \int e^x dx = e^x + C$
- Now, use the initial condition $y(0) = 2$:
- $2 = e^0 + C$
- $2 = 1 + C$
- $C = 1$
- So, the particular solution is: $y = e^x + 1$

3) Solving a simple First-Order ODE: $\frac{dy}{dx}+3y=6$

- Identify the integrating factor:
- The equation is in the form $\frac{dy}{dx}+P(x)y=Q(x)$, where $P(x)=3$ and $Q(x)=6$.
- The integrating factor, $\mu(x)$, is found using: $\mu(x)=e^{\int P(x)dx}=e^{\int 3dx}=e^{3x}$
- Multiply the entire equation by the integrating factor: $e^{3x}\frac{dy}{dx}+3e^{3x}y=6e^{3x}$
- Rewrite the left side as a derivative:
- Notice that the left side of the equation is the derivative of $y\cdot e^{3x}$: $\frac{d}{dx}\left(y\cdot e^{3x}\right)=6e^{3x}$
- Integrate both sides: $\int\frac{d}{dx}\left(y\cdot e^{3x}\right)dx=\int 6e^{3x}dx$

    The left side integrates to: $y\cdot e^{3x}$

    The right side integrates to: $\int 6e^{3x}dx=6\int e^{3x}dx=6\cdot\frac{e^{3x}}{3}=2e^{3x}$
- So we have: $y\cdot e^{3x}=2e^{3x}+C$
- Solve for y : $y=\frac{2e^{3x}+C}{e^{3x}}=2+Ce^{-3x}$ where C is the constant of integration.
- Final solution: $y=2+Ce^{-3x}$ (This is the general solution)

9-2. Separable Differential Equation

Separable differential equations are a class of ordinary differential equations (ODEs) where the equation can be expressed as the product of two functions, each depending solely on one variable. This allows the equation to be rearranged such that all terms involving one variable are on one side of the equation, and all terms involving the other variable are on the other side.

These equations are particularly useful because they can often be solved by direct integration of both sides.

Formulas:

A first-order separable ODE can typically be written in the form: $\frac{dy}{dx} = g(x)h(y)$

This can be rearranged as: $\frac{1}{h(y)}dy = g(x)dx$

Integrating both sides then gives: $\int \frac{1}{h(y)}dy = \int g(x)dx$

Examples:

1) Solve $\frac{dy}{dx} = xy$ **:**

- This equation is separable and can be rewritten as: $\frac{1}{y}dy = xdx$
- Integrating both sides: $\int \frac{1}{y}dy = \int xdx \Rightarrow \ln|y| = \frac{x^2}{2} + C$ (where C is the integration constant)
- Solving for y: $|y| = e^{\frac{x^2}{2}+C} \Rightarrow y = \pm e^C e^{\frac{x^2}{2}}$ (let $\pm e^C = C_1$, another constant)
- The general solution is: $y = C_1 e^{\frac{x^2}{2}}$ (where C_1 includes both the sign and magnitude)

2) Solve $\frac{dy}{dx}=\frac{2x}{3y^2}$:

- Rearranging terms: $3y^2dy=2xdx$
- Integrating both sides: $\int 3y^2dy=\int 2xdx \Rightarrow y^3=\frac{x^2}{3}+C$
- Solving for y: $y=\left(\frac{x^2}{3}+C\right)^{\frac{1}{3}}$
- The general solution is: $y=\left(\frac{x^2}{3}+C\right)^{\frac{1}{3}}$

3) Solve the differential equation: $\frac{dy}{dx}=3xy$

- Separate the variables: $\frac{1}{y}dy=3xdx$
- Integrate both sides: $\int\frac{1}{y}dy=\int 3xdx$
- $\ln|y|=\frac{3x^2}{2}+C$
- Exponentiate both sides: $\left|y\right|=e^{\frac{3x^2}{2}+C}$
- $y=\pm e^Ce^{\frac{3x^2}{2}}$
- Since y can be positive or negative, we write the general solution as:
- Let $C_1=e^C$: $y=\pm C_1e^{\frac{3x^2}{2}}$

4) Solve the initial value problem (IVP): $\frac{dy}{dx} = 2y\cos(x)$ **with the initial condition** $y(0) = 1$**.**

- Separate the variables: $\frac{1}{y}dy = 2\cos(x)dx$
- Integrate both sides: $\int \frac{1}{y}dy = \int 2\cos(x)dx$
- $\ln|y| = 2\sin(x) + C$
- Exponentiate both sides: $|y| = e^{2\sin(x)+C}$
- $y = \pm e^{C} e^{2\sin(x)}$
- Let $C_1 = e^{C}$: $y = C_1 e^{2\sin(x)}$
- Use the initial condition $y(0) = 1 : 1 = C_1 e^{2\sin(0)}$
- $1 = C_1 e^{0} \Rightarrow C_1 = 1$
- So, the particular solution (due to initial condition) is: $y = e^{2\sin(x)}$

9-3. Euler's Method for Approximating Differential Equations (AP Cal BC)

Euler's Method is a straightforward numerical technique for approximating solutions to ordinary differential equations (ODEs). It is especially useful when an analytical solution is difficult or impossible to find. The method progresses step by step, starting from an initial condition, and uses the derivative at each step to estimate the value of the function at the next step.

Procedure:

Given an initial value problem defined by the differential equation $\frac{dy}{dx} = f(x,y)$ with the initial condition $y(x_0) = y_0$, Euler's method approximates y at subsequent points by:

1. Choosing a step size h, which is the interval between points where the solution is approximated.

2. Using the derivative at the beginning of the interval to estimate the slope that is then used to find the function's value at the end of the interval.

3. Repeating the process for each step over the desired range.

Formulas:

The iterative formula used in Euler's method is:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

where:

- y_{n+1} is the estimated value of the function at x_{n+1}
- y_n is the estimated value at x_n
- $f(x_n, y_n)$ is the derivative of y with respect to x evaluated at x_n and y_n
- h is the step size
- $x_{n+1} = x_n + h$

Examples:

1) Approximate y for $\frac{dy}{dx}=x+y$ with $y(0)=1$ and $h=0.1$:

- Starting from $x_0=0$ and $y_0=1$,
- Calculate $f(0,1)=0+1=1$,
- Using Euler's method: $y_1=1+0.1\times1=1.1$
- Now, at $x_1=0.1$,
- Calculate $f(0.1,1.1)=0.1+1.1=1.2$,
- Next step: $y_2=1.1+0.1\times1.2=1.22$
- Continuing this process yields a step-by-step approximation of y at increasing values of x .

2) Use Euler's Method with a step size of $h=0.5$ to find the first approximation y_1 of the differential equation $\frac{dy}{dx}=3y$ given that $y(0)=1$.

1. Initial values: $x_0=0,\quad y_0=1$
Step size: $h=0.5$

2. Euler's formula: $y_{n+1}=y_n+h\cdot f(x_n,y_n)$
where $f(x,y)=3y$.

3. Calculate the first approximation:
- For $n=0$: $y_1=y_0+h\cdot(3y_0)=1+0.5\cdot(3\cdot1)=2.5$

4. Result:
The first approximation y_1 is $y_1=2.5$.

3) Apply Euler's Method with $h = 0.2$ to estimate y_1 for $\frac{dy}{dx} = y - x$ starting from $x_0 = 0$, $y_0 = 2$.

1. Initial values: $x_0 = 0, \quad y_0 = 2$
Step size: $h = 0.2$

2. Euler's formula: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
where $f(x,y) = y - x$.

3. Calculate the first approximation:
- For $n = 0$: $y_1 = y_0 + h \cdot (y_0 - x_0) = 2 + 0.2 \cdot (2 - 0) = 2.4$

4. Result:
The first approximation y_1 is $y_1 = 2.4$.

9-4. Exponential Growth and Decay Model by Differential Equations

Exponential growth and decay are processes that increase or decrease at rates proportional to their current size. This behavior can be modeled using differential equations, making it a fundamental concept in biology (population dynamics), physics (radioactive decay), economics (compound interest), and more.

The basic premise is that the rate of change of a quantity y over time t is proportional to the quantity itself.

Formulas:

The differential equation that models exponential growth or decay is:

$$\frac{dy}{dt} = ky$$

where:

- y is the quantity of interest,
- k is a constant that determines the rate and direction of growth or decay:
- $k > 0$ leads to exponential growth,
- $k < 0$ leads to exponential decay.

General Solutions:

The general solution to the exponential growth/decay equation is:

$$y(t) = y_0 e^{kt}$$

where:

- y_0 is the initial value of y at $t = 0$,
- e is the base of the natural logarithm.

Examples:

1) Exponential Growth: Suppose a population of bacteria doubles every hour. If the initial population is 100, model the population after 3 hours.

- Let $y(t)$ be the population at time t, with $y(0)=100$.
- Since the population doubles every hour, k corresponds to the natural logarithm of 2(since $e^{kt}=2$ when $t=1$).
- Thus, $k=\ln(2)$.
- The population model is $y(t)=100e^{\ln(2)t}$.
- To find the population after 3 hours, $y(3)=100\cdot 2^3=800$.

2) Exponential Decay: Consider a radioactive substance with a half-life of 3 hours. If the initial mass is 200 grams, find the mass remaining after 9 hours.

- Let $y(t)$ be the mass at time t, with $y(0)=200$.
- For decay, k is negative and $e^{kt}=0.5$ when $t=3$, so $k=-\ln(2)/3$.
- The mass model is $y(t)=200e^{-\ln(2)t/3}$.
- After 9 hours, $y(9)=200\cdot 0.5^3=25$ grams.

3) Solve the differential equation for a population experiencing exponential growth at a rate of 20% per year with an initial population of 500.

- Given the equation $\frac{dP}{dt}=0.2P$, solving with $P(0)=500$: $P(t)=500e^{0.2t}$

4) A cooling body follows Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - T_{\text{env}})$, where T is the temperature of the body, T_{env} is the environmental temperature, and k is a positive constant. If a body cools from 100°C to 70°C in 10 minutes in a room at 20°C, find the temperature of the body after another 10 minutes.

- Solve differential equation: $\int \frac{1}{T - T_{\text{env}}} dT = -k \int dt$
- First, find k using the initial condition and solve: $T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{-kt}$
- $T(t) - 20 = (100 - 20)e^{-kt}$
- Given $T(0) = 100$, $T(10) = 70$, and $T_{\text{env}} = 20$, set up the equation and solve for k, then use it to find $T(20)$
- $70 - 20 = 80e^{-10k} \Rightarrow k = -\frac{1}{10}\ln\left(\frac{5}{8}\right)$
- $T(20) - 20 = 80e^{-k \cdot 20} \Rightarrow T(20) = 51.25^{\circ}\text{C}$

9-5. Radioactive Decay Model by Differential Equations

Radioactive decay is a fundamental process in nuclear physics where unstable atomic nuclei lose energy by emitting radiation in the form of particles or electromagnetic waves. This decay occurs at a rate proportional to the current amount of the substance, which is a perfect scenario for modeling with differential equations.

Formulas:

The differential equation that models radioactive decay is:

$$\frac{dN}{dt} = -kN$$

where:

- N is the number of undecayed nuclei at time t ,

- k is the decay constant, specific to each radioactive material.

The negative sign indicates that the number of nuclei decreases over time.

Solutions:

The general solution to this differential equation is:

$$N(t) = N_0 e^{-kt}$$

where:

- N_0 is the initial number of nuclei at $t = 0$.

Decay Constant and Half-Life:

- The half-life ($t_{1/2}$) of a radioactive substance is the time it takes for half of the radioactive nuclei to decay.
- The relationship between the decay constant and half-life is:

$$t_{1/2} = \frac{\ln 2}{k}$$

Examples:

1) Modeling Radioactive Decay of Carbon-14:

- Carbon-14 has a half-life of approximately 5730 years.
- Using the relationship between half-life and decay constant:

$$k = \frac{\ln 2}{5730} \approx 0.000121 \text{ per year}$$

- If a sample initially contains $N_0 = 1000$ atoms of Carbon-14, the number of atoms remaining after t years is:

$$N(t) = 1000 \cdot e^{-0.000121t}$$

- To find the remaining number of Carbon-14 atoms after 11,460 years (two half-lives):

$$N(11460) = 1000 \cdot e^{-0.000121 \times 11460} \approx 250 \text{ atoms}$$

2) Calculating Remaining Radioactive Substance:

- Suppose a substance has a half-life of 3 hours, and the initial mass is 200 grams.
- The decay constant is:

$$k = \frac{\ln 2}{3}$$

- After 9 hours, the mass remaining is:

$$m(9) = 200 \cdot e^{-\frac{\ln 2}{3} \times 9} = 200 \cdot e^{-3\ln 2} = 200 \cdot \left(\frac{1}{2}\right)^3 = 25 \text{ grams}$$

3) Write the differential equation representing the decay of a substance at a rate of 5% per year. If you start with 200 grams, what is the initial condition?

- The differential equation is $\frac{dN}{dt} = -0.05N$.
- The initial condition is $N(0) = 200$ grams.

4) Given the radioactive decay model $\frac{dN}{dt} = -0.04N$ and an initial amount of 150 grams, find the amount of the substance remaining after 10 years.

- This is a first-order linear differential equation, and its solution is: $N(t) = N_0 e^{-kt}$
- Given: k = 0.04
  - Initial amount $N_0 = 150$ grams
  - Time $t = 10$ years
- Solve the equation $N(t) = 150e^{-0.04t}$.
- After 10 years: $N(10) \approx 150 \cdot 0.67032$
- $N(10) \approx 100.548$grams

9-6. Newton's Law of Cooling Model by Differential Equations

Newton's Law of Cooling describes the rate at which an object cools or heats towards the ambient temperature. It assumes that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature, a principle that is applicable in various scientific fields such as physics, engineering, and meteorology.

Formulas:

The differential equation that models Newton's Law of Cooling is given by:

$$\frac{dT}{dt} = -k(T - T_a)$$

where:

- $T(t)$ is the temperature of the object at time t ,
- T_a is the ambient temperature, which is assumed to be constant,
- k is the positive constant of proportionality that depends on the characteristics of the object and its environment.

Solutions:

The solution to this differential equation, assuming $T(0) = T_0$ (the initial temperature of the object), is:

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

This formula shows that as t approaches infinity, $T(t)$ approaches T_a, indicating that the object's temperature eventually stabilizes at the ambient temperature.

Examples:

1) Cooling of a Cup of Coffee:

- Suppose a cup of coffee initially at 90°C is left in a room where the ambient temperature is 20°C. If the coffee cools to 70°C in 10 minutes, find the temperature after 30 minutes.

- First, determine k using the data provided:

$$70 = 20 + (90 - 20)e^{-10k}$$
$$50 = 70e^{-10k}$$
$$e^{-10k} = \frac{50}{70} \approx 0.714$$
$$-10k = \ln(0.714)$$
$$k \approx \frac{-\ln(0.714)}{10} \approx 0.0351 \text{ per minute}$$

- Use k to find the temperature after 30 minutes:

$$T(30) = 20 + 70e^{-0.0351 \times 30}$$
$$T(30) \approx 20 + 70 \times 0.354 \approx 44.78^{\circ} C$$

- Therefore, the coffee will be approximately 44.78°C after 30 minutes.

2) Cooling of Heated Metal in a Cooler Environment:

- A piece of metal is heated to 150°C and then placed in a cooler environment with a constant temperature of 25°C. Assuming $k = 0.1$ per minute, calculate the metal's temperature after 20 minutes.

- A piece of metal is heated to 150°C and then placed in a cooler environment with a constant temperature of 25°C. Assuming $k = 0.1$ per minute, calculate the metal's temperature after 20 minutes.

- Applying the cooling formula:

$$T(20) = 25 + (150 - 25)e^{-0.1 \times 20}$$
$$T(20) = 25 + 125e^{-2}$$
$$T(20) = 25 + 125 \times 0.1353 \approx 41.9125^{\circ} C$$

- The metal will be approximately 41.91°C after 20 minutes.

3) Write the differential equation for an object heating up from 50°C in a room where the temperature is 70°C. The constant of proportionality k is 0.1.

- The differential equation is $\frac{dT}{dt} = -0.1(T - 70)$.
- The negative sign indicates cooling, but since the object's temperature is below the ambient, it will actually warm up.

4) Given an initial object temperature of 100°C and an ambient temperature of 20°C with $k = 0.05$, find the temperature of the object after 10 minutes.

- The differential equation is: $\frac{dT}{dt} = 0.05(20 - T)$
- Use the law of cooling: $T(t) = 20 + (100 - 20)e^{-0.05 \cdot t}$
- At $t = 10$ minutes: $T(10) = 20 + 80e^{-0.5}$ $T(10) \approx 68.52^{\circ}\text{C}$