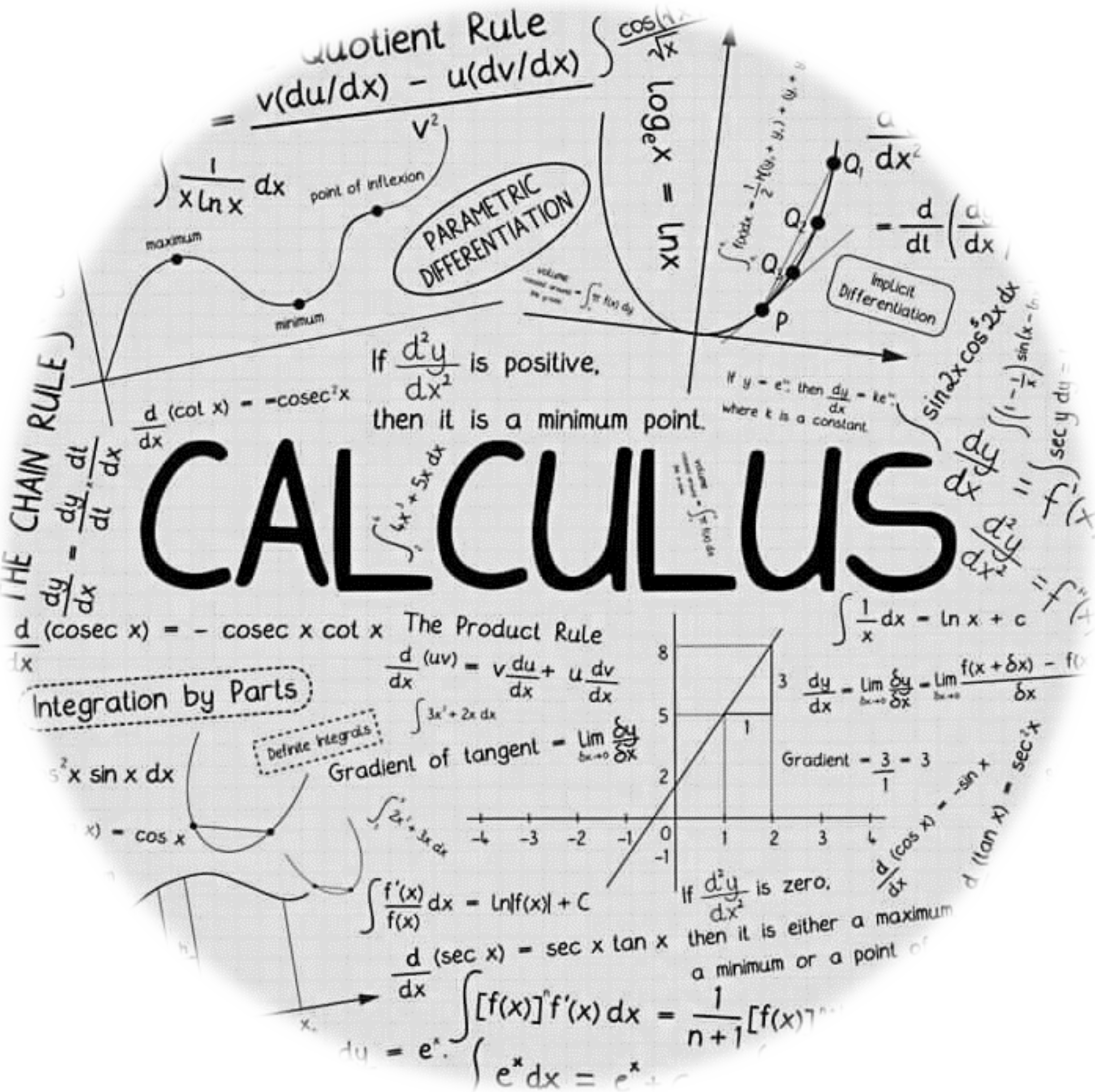
eSpyMath: AP Calculus AB

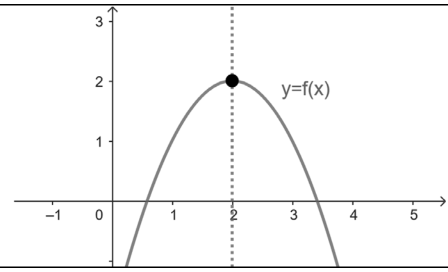
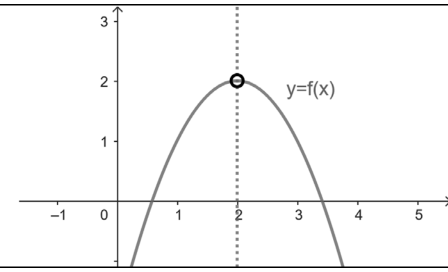
Contents

Chapter 0. Foundations

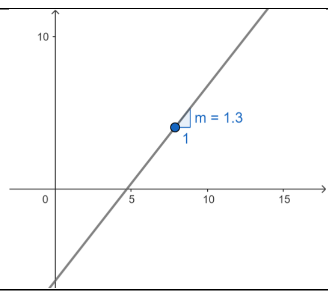
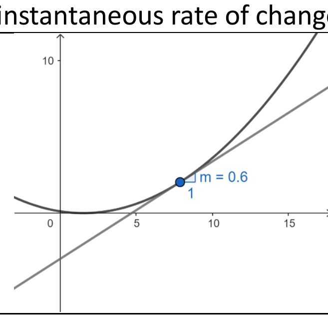


0-1. Why Calculus

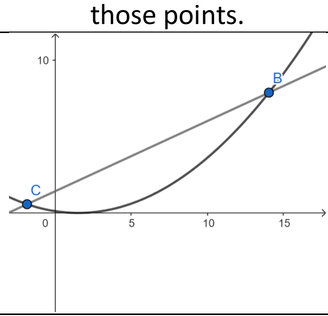
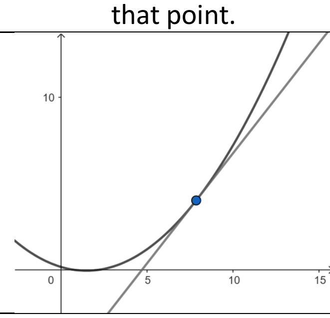
1. Value of $f(x)$ when $x = c$:

Without Calculus	With Differential Calculus
You directly find the value of the function f at $x = c$.	You consider the limit of $f(x)$ as x approaches c, which can be more precise, especially if f is not continuous at c.
	

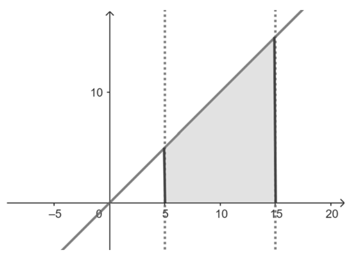
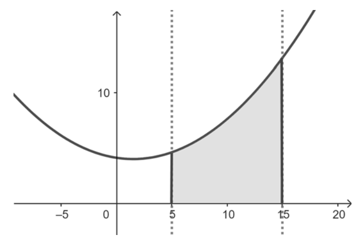
2. Slope of a line:

Without Calculus	With Differential Calculus
The slope is the change in y divided by the change in x ($\Delta y / \Delta x$).	**The slope of a curve at a point** is found using the derivative (dy / dx), representing the instantaneous rate of change.
	

3. Secant line to a curve:

Without Calculus	With Differential Calculus
A secant line intersects the curve at two points, representing the average rate of change between those points.	**A tangent line** touches the curve at one point, representing the instantaneous rate of change at that point.
	

4. Area under the line or curve:

Without Calculus	With Differential Calculus
You find the area by multiplying the length and width of the rectangle/polygon.	**You find the area under a curve** using integration, which can handle more complex shapes.
	

5. Length of a line segment:

Without Calculus	With Differential Calculus
The length is the distance between two points.	You find the **length of an arc** (curved line) using integration.

6. Surface area of a cylinder:

Without Calculus	With Differential Calculus
You calculate the surface area using the formula for a cylinder.	You find **the surface area of a solid** of revolution using integration, which can handle more complex shapes.

7. Mass of a solid of constant density:

Without Calculus	With Differential Calculus
The mass is found by multiplying the volume by the constant density.	You calculate the **mass of a solid** with variable density using integration.

8. Volume of a rectangular solid:

Without Calculus	With Differential Calculus
The volume is found by multiplying length, width, and height.	You find **the volume of a region under a surface** using integration.

0-2. Increasing & Decreasing functions (use in curve sketching)

1) Determine the domain and range of the function $g(x) = 2e^x$.	2) Investigate the properties of the function $h(x) = e^{x-2}$. Specifically, determine the domain, range, and whether the function is increasing or decreasing.
3) Determine the domain and range of the function $g(x) = 2\sin(x)$.	4) Investigate the properties of the function $h(x) = \sin(x - \frac{\pi}{4})$. Specifically, determine the domain, range, and symmetry.
5) Determine the domain and range of the function $g(x) = \frac{1}{2x}$.	6) Investigate the properties of the function $h(x) = \frac{1}{x-3}$. Specifically, determine the domain, range, and symmetry.

Solutions:

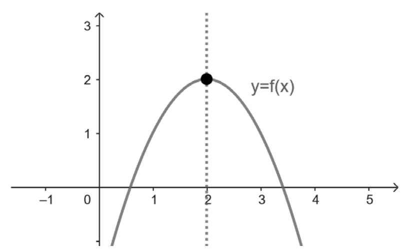
1) Determine the domain and range of the function $g(x) = 2e^x$. The function $g(x) = 2e^x$ is a transformation of the basic exponential function e^x. - Domain: The domain of e^x is all real numbers, $(-\infty, \infty)$. Since $g(x) = 2e^x$ is just a vertical stretch, the domain remains the same. - Domain: $(-\infty, \infty)$ - Range: The range of . e^x . is $(0, \infty)$. Multiplying by 2 stretches the range but does not change its lower or upper bounds. - Range: $(0, \infty)$	**2) Investigate the properties of the function $h(x) = e^{x-2}$. Specifically, determine the domain, range, and whether the function is increasing or decreasing.** The function $h(x) = e^{x-2}$ is a horizontal shift of the basic exponential function e^x. - Domain: The domain of e^x is all real numbers, $(-\infty, \infty)$. The horizontal shift does not affect the domain. - Domain: $(-\infty, \infty)$ - Range: The range of e^x is $(0, \infty)$. The horizontal shift does not affect the range. - Range: $(0, \infty)$ - Increasing/Decreasing: The function e^x is always increasing, and the horizontal shift does not change this property. - Increasing on $(-\infty, \infty)$
3) Determine the domain and range of the function $g(x) = 2\sin(x)$. The function $g(x) = 2\sin(x)$ is a vertical stretch of the basic sine function $\sin(x)$. - Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty, \infty)$. The vertical stretch does not affect the domain. - Domain: $(-\infty, \infty)$ - Range: The range of $\sin(x)$ is $[-1, 1]$. Multiplying by 2 stretches the range to $[-2, 2]$. - Range: $[-2, 2]$	**4) Investigate the properties of the function $h(x) = \sin(x - \frac{\pi}{4})$. Specifically, determine the domain, range, and symmetry.** The function $h(x) = \sin(x - \frac{\pi}{4})$ is a horizontal shift of the basic sine function $\sin(x)$. - Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty, \infty)$. The horizontal shift does not affect the domain. - Domain: $(-\infty, \infty)$ - Range: The range of $\sin(x)$ is $[-1, 1]$. The horizontal shift does not affect the range. - Range: $[-1, 1]$ - Symmetry: The function $\sin(x)$ is an odd function, symmetric about the origin. However,

	$h(x)=\sin(x-\frac{\pi}{4})$ is not an odd/even function since it is a horizontally shifted sine function. - The sine function has a period of 2π, so the function $h(x)=\sin\left(x-\frac{\pi}{4}\right)$ also has a period of 2π.
5) Determine the domain and range of the function $g(x)=\frac{1}{2x}$. The function $g(x)=\frac{1}{2x}$ is a vertical compression of the basic function $\frac{1}{x}$. - - Domain: The domain of $\frac{1}{x}$ is all real numbers except $x=0$, because division by zero is undefined. The vertical compression does not affect the domain. - Domain: $(-\infty,0)\cup(0,\infty)$ - - Range: The range of $\frac{1}{x}$ is all real numbers except $y=0$, since $\frac{1}{x}$ never equals zero. The vertical compression does not affect the range. - Range: $(-\infty,0)\cup(0,\infty)$	**6) Investigate the properties of the function $h(x)=\frac{1}{x-3}$. Specifically, determine the domain, range, and symmetry.** The function $h(x)=\frac{1}{x-3}$ is a horizontal shift of the basic function $\frac{1}{x}$. - Domain: The domain of $\frac{1}{x}$ is all real numbers except $x=0$. For $\frac{1}{x-3}$, the horizontal shift means the function is undefined at $x=3$. - Domain: $(-\infty,3)\cup(3,\infty)$ - Range: The range of $\frac{1}{x}$ is all real numbers except $y=0$. The horizontal shift does not affect the range. - Range: $(-\infty,0)\cup(0,\infty)$ - Symmetry: The function $\frac{1}{x}$ is an odd function, symmetric about the origin. However, the horizontal shift breaks this symmetry. - Symmetry: None

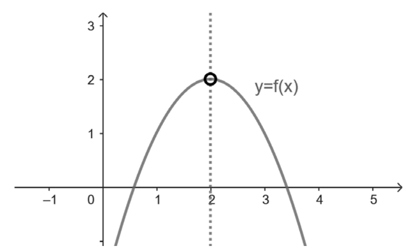
0-3. Limit Foundation

Find the limit?

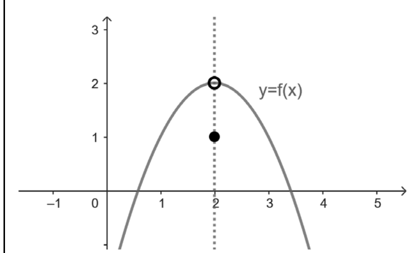
1) $\lim_{x \to 2} f(x)$



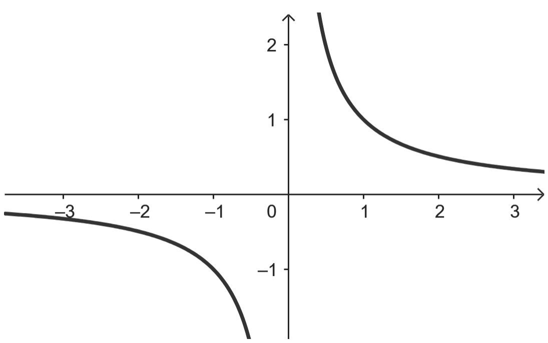
2) $\lim_{x \to 2} f(x)$



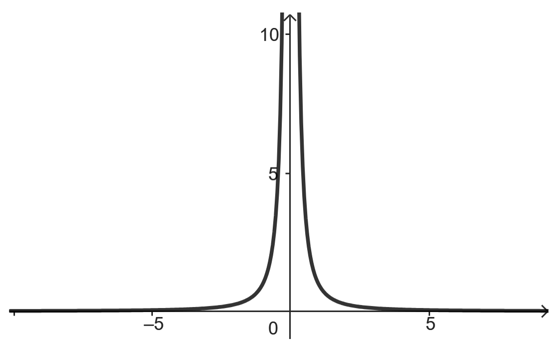
3) $\lim_{x \to 2} f(x)$



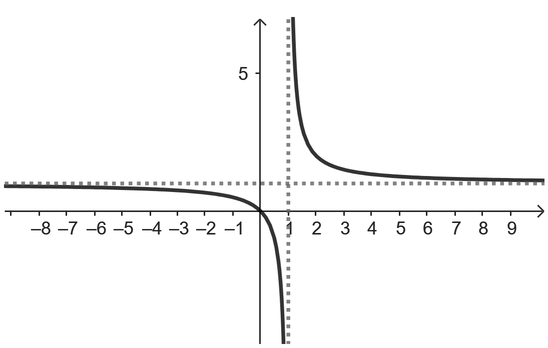
4) $\lim_{x \to 0^-} \frac{1}{x} =$ 5) $\lim_{x \to 0^+} \frac{1}{x} =$



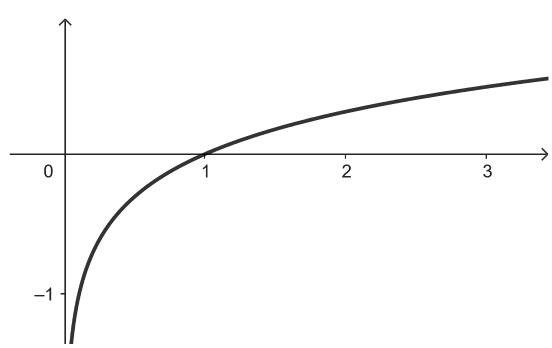
6) $\lim_{x \to 0^-} \frac{1}{x^2} =$ 7) $\lim_{x \to 0^+} \frac{1}{x^2} =$



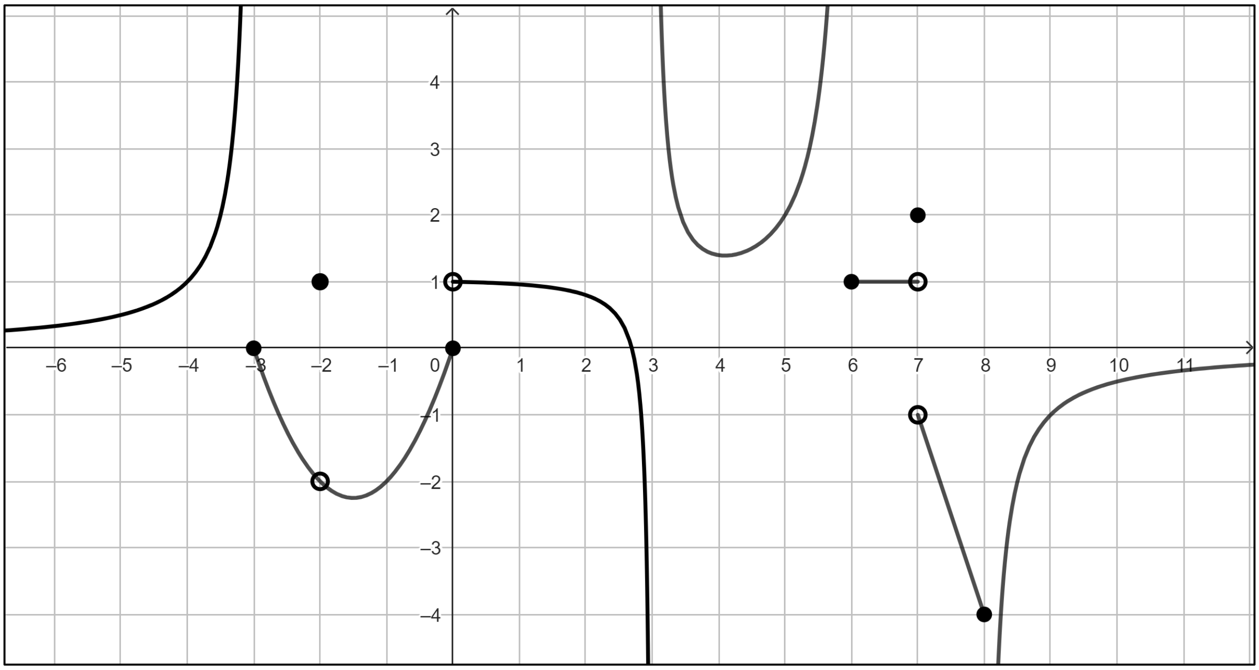
8) $\lim_{x \to 1^-} \frac{x}{x-1} =$ 9) $\lim_{x \to 1^+} \frac{x}{x-1} =$



10) $\lim_{x \to \infty} \log(x) =$ 11) $\lim_{x \to 0^+} \log(x) =$



12) Given the graph of $f(x)$ above, find the limit.



$\lim_{x\to-\infty} f(x)=$	$\lim_{x\to\infty} f(x)=$	$\lim_{x\to-3^+} f(x)=$	$\lim_{x\to-3^-} f(x)=$
$\lim_{x\to-2^-} f(x)=$	$\lim_{x\to-2^+} f(x)=$	$\lim_{x\to0^+} f(x)=$	$\lim_{x\to0^-} f(x)=$
$\lim_{x\to3^+} f(x)=$	$\lim_{x\to3^-} f(x)=$	$\lim_{x\to6^+} f(x)=$	$\lim_{x\to6^-} f(x)=$
$\lim_{x\to7^+} f(x)=$	$\lim_{x\to7^-} f(x)=$	$\lim_{x\to8^+} f(x)=$	$\lim_{x\to8^-} f(x)=$
$f(-2)=$	$f(7)=$		

Solutions:

1) $\lim_{x\to 2} f(x)=2$	2) $\lim_{x\to 2} f(x)=2$	3) $\lim_{x\to 2} f(x)=2$

4) $\lim_{x\to 0^-} \frac{1}{x}=-\infty$	5) $\lim_{x\to 0^+} \frac{1}{x}=\infty$	6) $\lim_{x\to 0^-} \frac{1}{x^2}=\infty$	7) $\lim_{x\to 0^+} \frac{1}{x^2}=\infty$

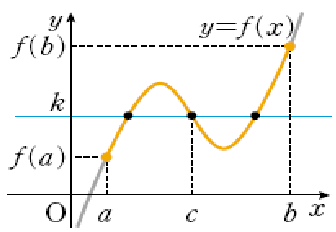
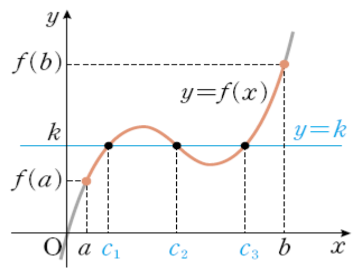
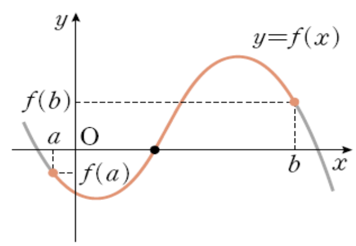
8) $\lim_{x\to 1^-} \frac{x}{x-1}=-\infty$	9) $\lim_{x\to 1^+} \frac{x}{x-1}=\infty$	10) $\lim_{x\to\infty} \log(x)=0$	11) $\lim_{x\to 0^+} \log(x)=-\infty$

12)			
$\lim_{x\to -\infty} f(x)=0$	$\lim_{x\to\infty} f(x)=0$	$\lim_{x\to -3^+} f(x)=0$	$\lim_{x\to -3^-} f(x)=\infty$
$\lim_{x\to -2^-} f(x)=-2$	$\lim_{x\to -2^+} f(x)=-2$	$\lim_{x\to 0^+} f(x)=1$	$\lim_{x\to 0^-} f(x)=0$
$\lim_{x\to 3^+} f(x)=\infty$	$\lim_{x\to 3^-} f(x)=-\infty$	$\lim_{x\to 6^+} f(x)=1$	$\lim_{x\to 6^-} f(x)=-\infty$
$\lim_{x\to 7^+} f(x)=-1$	$\lim_{x\to 7^-} f(x)=1$	$\lim_{x\to 8^+} f(x)=-\infty$	$\lim_{x\to 8^-} f(x)=-4$
$f(-2)=1$	$f(7)=2$		

0-4. Basic Theorems

Theorem: Intermediate Value Theorem, IVT

If **f is continuous** on a closed interval [a, b] and k is any number between f(a) and f(b), then there exists at least one number c in (a, b) such that f(c) = k.

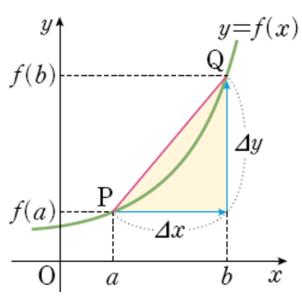
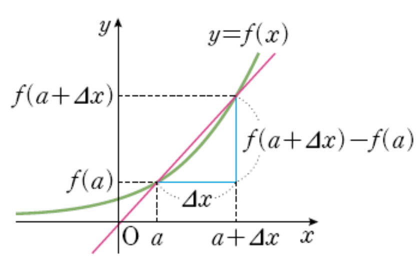
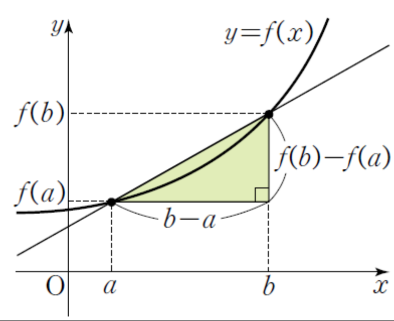
		If f(b)>0 and f(a)<0, then there exists at least one solution.
		

Definition of the Average Rate of Change

The average rate of change of y (slope m) with respect to x over the interval $[a,b]$ is given by:

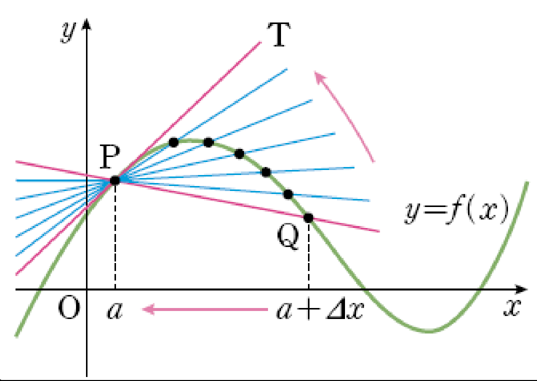
$$m=\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}=\frac{f(a+\Delta x)-f(a)}{\Delta x}=\frac{f(a+h)-f(a)}{h}$$

where $h=\Delta x=b-a$.

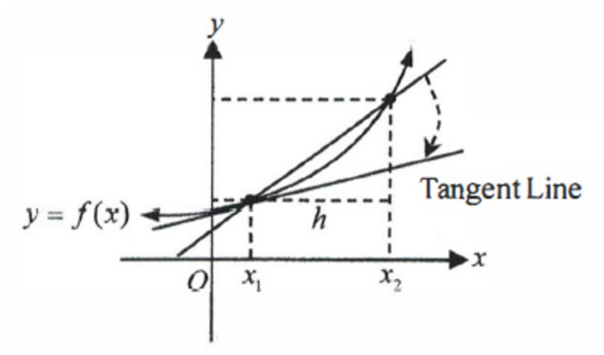
		

Definition of the Instant Rate of Change

The graph demonstrates the concept of secant lines approaching the tangent line at a specific point on a curve as the interval between the points on the secant line (Δx) approaches zero. This visual representation helps in understanding the definition of the derivative, which is the slope of the tangent line at a given point on the function.



$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$



$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- **Secant Lines**: The lines passing through points P and Q are secant lines. These lines intersect the curve at two points and approximate the slope of the function between those points.

  - As Δx decreases (meaning Q moves closer to P), the secant lines approach the slope of the tangent line at P.

- **Tangent Line T**:

  - Definition: The tangent line at point P is the line that just touches the curve at P without crossing it. This line represents the instantaneous rate of change of the function at $x = a$.

  - Slope of the Tangent Line: The slope of the tangent line at P is the limit of the slopes of the secant lines as Δx approaches zero.

- The slope of the tangent line at $x = a$ is given by the derivative: (replace Δx to h)

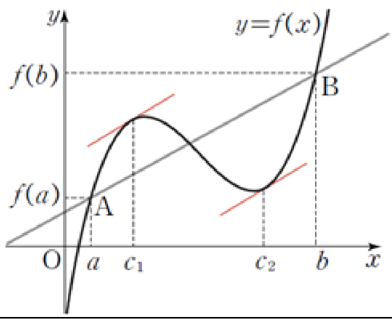
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

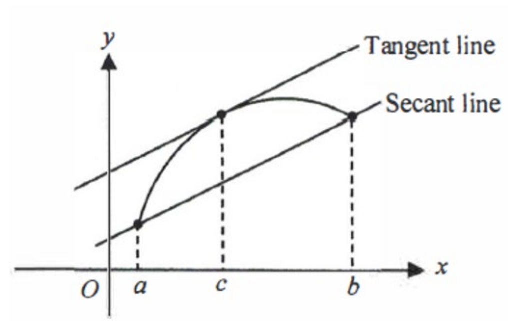
THEOREM: The Mean-Value Theorem (MVT)

If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c between a and b such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

The slope of the secant line is equal to the slope of the tangent line.





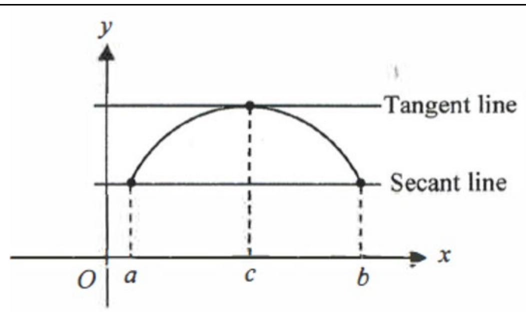
The Mean-Value Theorem guarantees that there is at least one point c in the interval (a, b) where the tangent line has the same slope as the secant line.

THEOREM: Rolle's Theorem (Special case of the Mean Value Theorem)

Let f be continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

If $f(b)=f(a)$, then there exists at least one number c between a and b such that

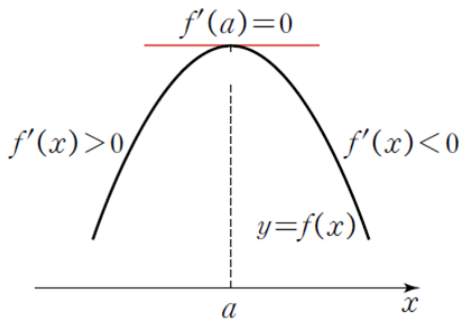
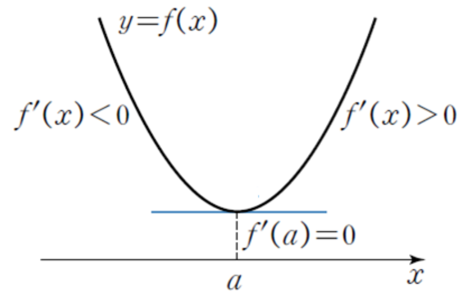
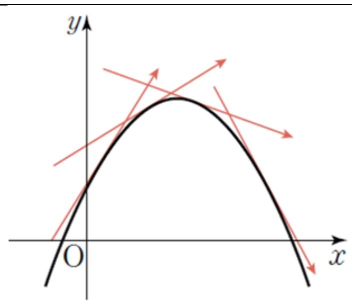
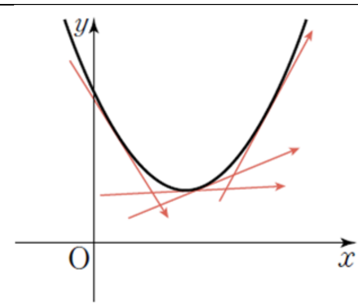
$$\frac{f(b)-f(a)}{b-a}=f'(c)=0.$$



Rolle's Theorem will guarantee the existence of an extreme value (relative maximum or relative minimum) in the interval.

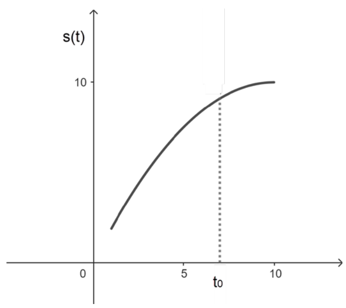
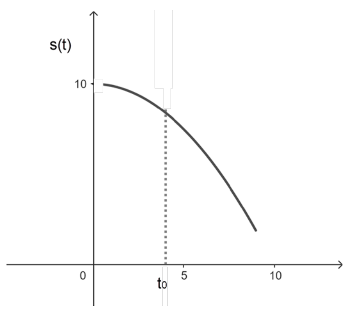
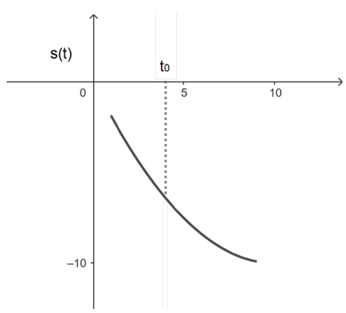
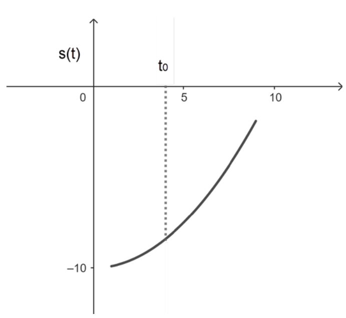
Maxima and Minima

- Maxima: At a local maximum, the derivative changes from positive to negative.
- Minima: At a local minimum, the derivative changes from negative to positive.
- Tangent lines at the critical points where $f'(a)=0$ confirm the behavior of the slopes, showing a peak for maxima and a valley for minima.

Maxima (relative maximum at x=a)	**Minima (relative minimum at x=a)**
	
- Function $y=f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$. - Left of a: $f'(x)>0$, the function is increasing. - Right of a: $f'(x)<0$, the function is decreasing. - This indicates a local maximum at $x=a$.	- Function $y=f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$. - Left of a: $f'(x)<0$, the function is decreasing. - Right of a: $f'(x)>0$, the function is increasing. - This indicates a local minimum at $x=a$.
	
- Shows the tangent lines with positive slopes approaching $x=a$ from the left and negative slopes after $x=a$, confirming the maximum. - Concave Downward	- Shows the tangent lines with negative slopes approaching $x=a$ from the left and positive slopes after $x=a$, confirming the minimum. - Concave Upward

0-5. Behavior of the Particle about Position vs. Time Curve

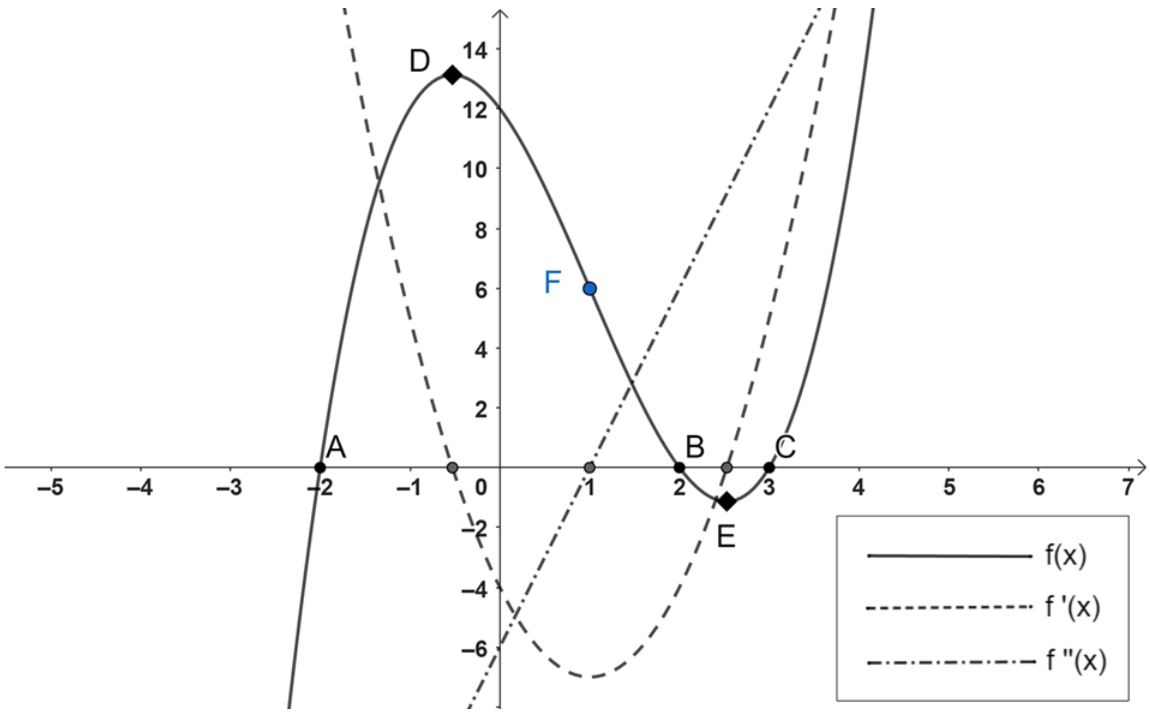
Observe behavior of the particle about the position versus time curve.

	At $t = t_0$ - Curve has positive slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is decreasing. - Particle is slowing down. - $v(t_0) > 0$ and $a(t_0) < 0$
	2. At $t = t_0$ - Curve has negative slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the negative direction. - Velocity is decreasing. - Particle is speeding up. - $v(t_0) < 0$ and $a(t_0) < 0$
	3. At $t = t_0$ - Curve has negative slope. - Curve is concave up. - $s(t_0) < 0$ - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the negative side of the origin. - Particle is moving in the negative direction. - Velocity is increasing. - Particle is slowing down. - $v(t_0) < 0$ and $a(t_0) > 0$
	4. At $t = t_0$ - Curve has positive slope. - Curve is concave up. - $s(t_0) < 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is increasing. - Particle is speeding up. - $v(t_0) > 0$ and $a(t_0) > 0$

0-6. Derivative Test

Concept Expansion from Pre-Calculus:

Can you find local maximum (D) and local minimum (E) values and the inflection point (F) for $f(x) = x^3 - 3x^2 - 4x + 12$ without using Calculus Concept?



Solution)

- We can easily find roots (A, B & C) by factorization $\Rightarrow roots: x = -2, 2, 3$
- However, it is not easy to find x values for D (local max), E(local min) and F(Inflection point). Before calculus, to solve this problem, we may need to use the approximation method.
- Once we learn about derivatives, then we can find these points easily.

Practice Example: Sketch the polynomial graph of $f(x)=x^3-3x^2-24x+32$ **by using** $f'(x)$**,** $f''(x)$

Solution Steps:

1. Find $f(x)$, $f'(x)$, $f''(x)$

- $f(x)=x^3-3x^2-24x+32$
- $f'(x)=3x^2-6x-24$
- $f''(x)=6x-6$

2. Find the first derivative ($f'(x)$) equal to zero to find **critical points** and its functional values if exist

- $3x^2-6x-24=0 \Rightarrow 3(x-4)(x+2)=0 \Rightarrow x=4, x=-2$
- $f(-2)=(-2)^3-3(-2)^2-24(-2)+32=60$ (maxima)
- $f(4)=(4)^3-3(4)^2-24(4)+32=-48$ **(minima)**

3. Set the second derivative **(** $f''(x)$ **)** equal to zero to find **inflection points** and its functional values if exist

- $6x-6=0 \Rightarrow 6(x-1)=0 \Rightarrow x=1$
- $f(1)=(1)^3-3(1)^2-24(1)+32=6$ **(inflection point)**

4. Determine the y-intercept

- $f(0)=0^3-3(0)^2-24(0)+32=32$

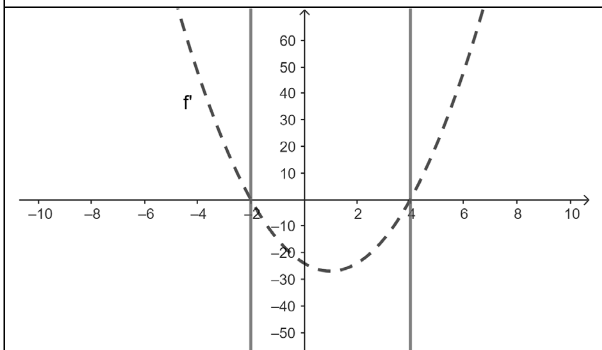
5. Determine the **concavity and relative extrema** using the first and second derivatives

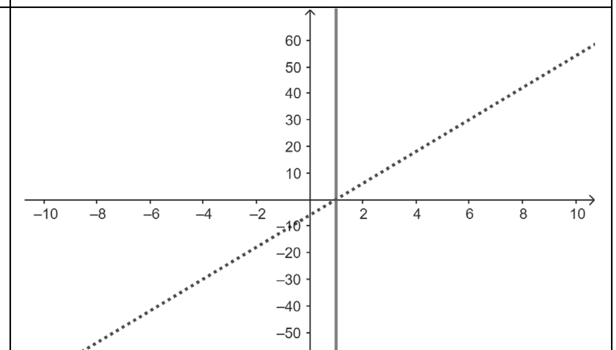
Number of critical points (including inflection points): 3, so need 4 sections on the graph

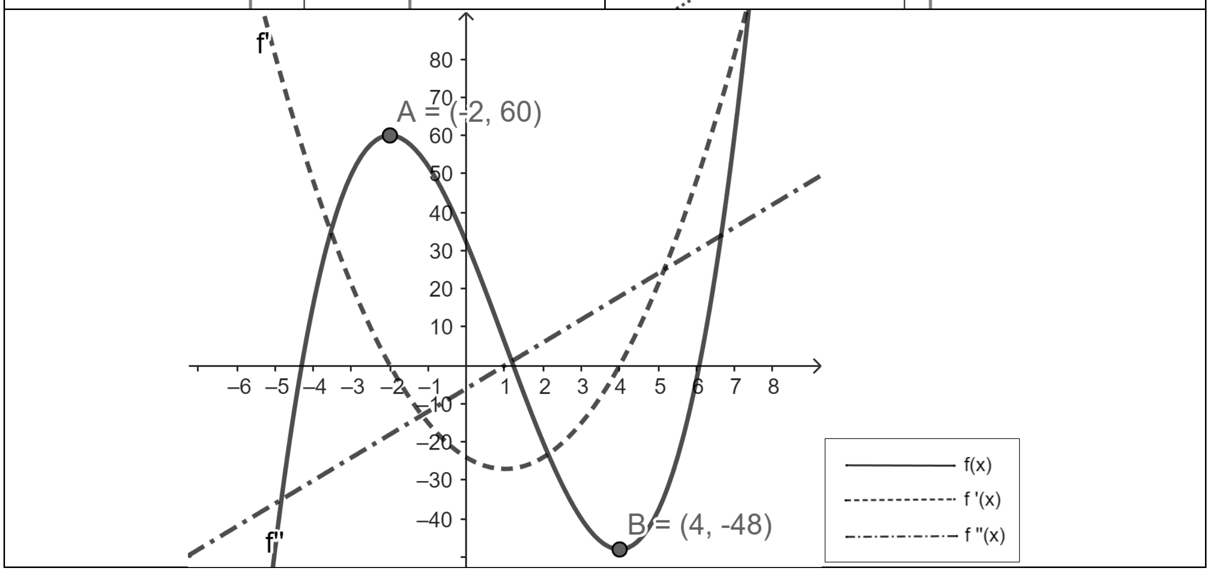
$f(x)=x^3-3x^2-24x+32$							
$f'(x)=3x^2-6x-24$	+	-2	-		-	4	+
$f''(x)=6x-6$	-		-	1	+		+

6. Sketch the graph:

$f'(x)$	$f''(x)$







1. **Plot the relative maximum at A** $(-2, 60)$.
2. **Plot the relative minimum at B** $(4, -48)$.
3. **Plot the inflection point at C** $(1, 6)$.
4. **Plot the y-intercept at D** $(0, 32)$.
5. **Draw the curve concave down from** $(-\infty, -2)$ **, then continue concave down through** $(-2, 60)$ **to** $(1, 6)$.
6. **Switch to concave up from** $(1, 6)$ **to** $(4, -48)$ **and continue concave up to** (∞, ∞).

Practice Example: Sketch the rational function graph of $f(x)=\frac{x^2-4x+3}{x}$ **by using** $f'(x)$, $f''(x)$

To sketch the rational function $f(x)=\frac{x^2-4x+3}{x}$ using its first and second derivatives, follow these steps:

1. Simplify the Function: $f(x)=\frac{x^2-4x+3}{x}=x-4+\frac{3}{x}$

2. Find Asymptotes

- Vertical Asymptote: Occurs where the denominator is zero: $x=0$
- Slanted (oblique) Asymptote: $y=x-4$

3. Find Intercepts

- x-intercepts: Set $f(x)=0$: $x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$, So, $x=1$ and $x=3$.
- y-intercept: Set $x=0$: The function is undefined at $x=0$, so there is no y-intercept.

4. Find Critical Points (First Derivative)

- Find the first derivative $f'(x)$: $f'(x)=\frac{d}{dx}\left(x-4+\frac{3}{x}\right)=1-\frac{3}{x^2}$
- Set $f'(x)=0 : 1-\frac{3}{x^2}=0 \;\; x=\pm\sqrt{3}$

5. Find Points of Inflection (Second Derivative)

- Find the second derivative $f''(x)$: $f''(x)=\frac{d}{dx}\left(1-\frac{3}{x^2}\right)=\frac{6}{x^3}$
- Set $f''(x)=0$: $\frac{6}{x^3}=0$
- There are no real solutions. So, there are no points of inflection.

6. Analyze Intervals of Increase/Decrease

- For $x>0$:

- $f'(x)=1-\frac{3}{x^2}$

- If $x > \sqrt{3}$, $f'(x) > 0$ (positive).

- If $0 < x < \sqrt{3}$, $f'(x) < 0$ (negative).

- For $x < 0$:

- $f'(x) = 1 - \frac{3}{x^2}$

- Always $f'(x) < 0$ (negative).

So, $x = -\sqrt{3}$ and $x = \sqrt{3}$ are **critical points:**

- Increasing on $(\sqrt{3}, \infty)$

- Decreasing on $(0, \sqrt{3})$ and $(-\infty, 0)$

7. Sketch the Graph

- Asymptotes:

- Vertical asymptote at $x = 0$

- Horizontal asymptote at $y = 1$

- Intercepts:

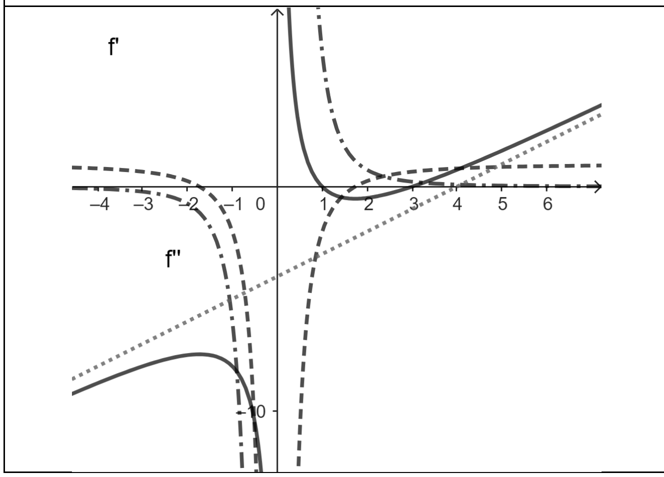
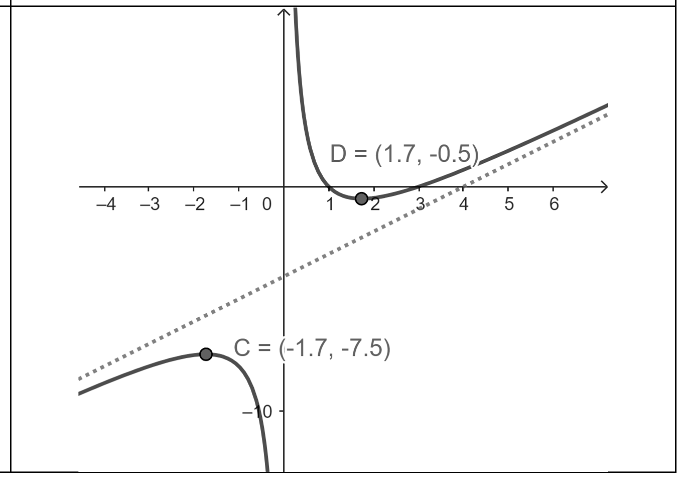
- x-intercepts at $x = 1$ and $x = 3$

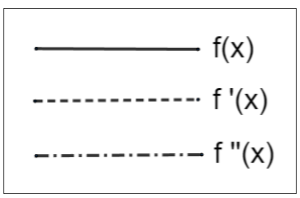
- Critical Points:

- Local Minimum at $x = \sqrt{3}$: $y = -0.5$

- Local Maximum at $x = -\sqrt{3}$: $y = -7.5$

$f(x) = \frac{x^2 - 3x - 4}{x}$							
$f'(x) = 1 - \frac{3}{x^2}$	+	$x = \sqrt{3}$ (local min)	-	0 (undefined)	-	$x = -\sqrt{3}$ (local max)	+
$f''(x) = \frac{6}{x^3}$	-	-	-	0 (undefined)	+	+	+

$f'(x) = 1 - \frac{3}{x^2},\ f''(x) = \frac{6}{x^3}$	$f(x) = \frac{x^2 - 3x - 4}{x}$
	

Legend: 

0-7. Derivative Formula

1. Derivative and Integral Rules

	Derivative	Integral (Antiderivative)
1	$\frac{d}{dx}n = 0$	$\int 0dx = C$
2	$\frac{d}{dx}x = 1$	$\int 1dx = x + C$
3	$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$	$\int \left[x^n\right]dx = \frac{x^{n+1}}{n+1} + C$
4	$\frac{d}{dx}\left[e^x\right] = e^x$	$\int \left[e^x\right]dx = e^x + C$
5	$\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$	$\int \left[\frac{1}{x}\right]dx = \ln x + C$
6	$\frac{d}{dx}\left[n^x\right] = n^x \ln n$	$\int \left[n^x\right]dx = \frac{n^x}{\ln n} + C$
7	$\frac{d}{dx}\left[\sin x\right] = \cos x$	$\int \left[\cos x\right]dx = \sin x + C$
8	$\frac{d}{dx}\left[\cos x\right] = -\sin x$	$\int \left[\sin x\right]dx = -\cos x + C$
9	$\frac{d}{dx}\left[\tan x\right] = \sec^2 x$	$\int \left[\sec^2 x\right]dx = \tan x + C$
10	$\frac{d}{dx}\left[\cot x\right] = -\csc^2 x$	$\int \left[\csc^2 x\right]dx = -\cot x + C$
11	$\frac{d}{dx}\left[\sec x\right] = \sec x \tan x$	$\int \left[\tan x \sec x\right]dx = \sec x + C$
12	$\frac{d}{dx}\left[\csc x\right] = -\csc x \cot x$	$\int \left[\cot x \csc x\right]dx = -\csc x + C$
13	$\frac{d}{dx}\left[\arcsin x\right] = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$
14	$\frac{d}{dx}\left[\arccos x\right] = -\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \arccos x + C$
15	$\frac{d}{dx}\left[\arctan x\right] = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \arctan x + C$

16	$\frac{d}{dx}[\text{arccot}\, x] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \text{arccot}\, x + C$
18	$\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \text{arcsec}\, x + C$
19	$\frac{d}{dx}[\text{arccsc}\, x] = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \text{arccsc}\, x + C$

2. General Differentiation Rules

Let **c** be a real number, **n** be a rational number, **u** and **v** be differentiable functions of **x**, let **f** be a differentiable function of **u**, and let **a** be a positive real number ($a \neq 1$).

	Rules	
1	Constant Rule	$\frac{d}{dx}[c] = 0$
2	Constant Multiple Rule	$\frac{d}{dx}[cu] = cu'$
3	Product Rule	$\frac{d}{dx}[uv] = uv' + vu'$
4	Chain Rule	$\frac{d}{dx}[f(u)] = f'(u)u'$
5	(Simple) Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$
6	Sum or Difference Rule	$\frac{d}{dx}[u \pm v] = u' \pm v'$
7	Quotient Rule	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
8	General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
9	Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$

		$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$
10	Derivatives of Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$ $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$ $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
11	Derivatives of Inverse Trigonometric Functions	$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arccot}\, x] = -\frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, x] = -\frac{1}{x\sqrt{x^2-1}}$
12	Derivatives of Inverse Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$ $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\lvert u \rvert \sqrt{u^2-1}}$

		$\frac{d}{dx}[\text{arccsc}u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$
13	Derivatives of Basic Hyperbolic Trigonometric Functions $\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\cosh(x) = \frac{e^x + e^{-x}}{2}$	$\frac{d}{dx}[\sinh(x)] = \cosh(x)$ $\frac{d}{dx}[\cosh(x)] = \sinh(x)$ $\frac{d}{dx}[\tanh(x)] = \text{sech}^2(x)$ $\frac{d}{dx}[\text{sech}(x)] = -\text{sech}(x)\tanh(x)$ $\frac{d}{dx}[\text{csch}(x)] = -\text{csch}(x)\coth(x)$ $\frac{d}{dx}[\coth(x)] = -\text{csch}^2(x)$
14	Derivatives of Inverse Hyperbolic Trigonometric Functions	. $\frac{d}{dx}[\sinh^{-1}(x)] = \frac{1}{\sqrt{x^2+1}}$. $\frac{d}{dx}[\cosh^{-1}(x)] = \frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$ $\frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$ $\frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$ $\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$
15	Derivatives of Exponential and Logarithmic Functions	$\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[\ln x] = \frac{1}{x}$ $\frac{d}{dx}[a^x] = (\ln a)a^x$ $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
16	Basic Differentiation Rules for Elementary Functions (**u & v** be differentiable functions of **x)**	$\frac{d}{dx}[u^n] = nu^{n-1}u'$ $\frac{d}{dx}[\lvert u\rvert] = \frac{u}{\lvert u\rvert}u', \quad u \neq 0$ $\frac{d}{dx}[\ln u] = \frac{u'}{u}$

		$\frac{d}{dx}[e^u]=e^u u'$ $\frac{d}{dx}[\log_a u]=\frac{u'}{(\ln a)u}$ $\frac{d}{dx}[a^u]=(\ln a)a^u u'$

3. Hyperbolic functions are analogs of the circular trigonometric functions, but for a hyperbola. They are extensively used in various areas of mathematics, including algebra, calculus, and complex analysis. Here are the basic hyperbolic functions along with their definitions:

1	Hyperbolic Sine ($\sinh x$)	$\sinh x=\frac{e^x-e^{-x}}{2}$
2	Hyperbolic Cosine ($\cosh x$)	$\cosh x=\frac{e^x+e^{-x}}{2}$
3	Hyperbolic Tangent ($\tanh x$)	$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$
4	Hyperbolic Cosecant ($\operatorname{csch} x$)	$\operatorname{csch} x=\frac{1}{\sinh x}=\frac{2}{e^x-e^{-x}}$
5	Hyperbolic Secant ($\operatorname{sech} x$)	$\operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}}$
6	Hyperbolic Cotangent ($\coth x$)	$\coth x=\frac{\cosh x}{\sinh x}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$

4. List of antiderivative formulas covering a wider range of functions. These include basic functions, exponential and logarithmic functions, trigonometric functions, and some of their inverses

	Functions	**Antiderivative formulas**
1	Constant Function	$\int a\,dx=ax+C$
2	Power Function	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C \quad (n\neq -1)$
3	Exponential Function	$\int e^x\,dx=e^x+C$

4	General Exponential Function	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$
5	Natural Logarithm	$\int \frac{1}{x}dx = \ln\|x\| + C$
6	Sine Functions	$\int \sin(x)dx = -\cos(x) + C$
7	Cosine Functions	$\int \cos(x)dx = \sin(x) + C$
8	Tangent Functions	$\int \tan(x)dx = -\ln\|\cos(x)\| + C$
9	Cotangent (cot) Functions	$\int \cot(x)dx = \ln\|\sin(x)\| + C$
10	Secant (sec) Functions	$\int \sec(x)dx = \ln\|\sec(x) + \tan(x)\| + C$
11	Cosecant (csc) Functions	$\int \csc(x)dx = -\ln\|\csc(x) + \cot(x)\| + C$
12	Inverse Sine (arcsin) Functions	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$
13	Inverse Tangent (arctan) Functions	$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$
14	sinh (Hyperbolic Sine) Functions	$\int \sinh(x)dx = \cosh(x) + C$
15	cosh (Hyperbolic Cosine) Functions	$\int \cosh(x)dx = \sinh(x) + C$
16	Integral of sec^2	$\int \sec^2(x)dx = \tan(x) + C$
17	Integral of csc^2	$\int \csc^2(x)dx = -\cot(x) + C$

0-8. Find Derivatives

1) Find the derivative of the function $f(x)=7$. (Constant Rule)	2) Find the derivative of the function $f(x)=5x^3$. (Constant Multiple Rule)
3) Find the derivative of the function $f(x)=x^2\sin(x)$. (Product Rule)	4) Find the derivative of the function $f(x)=\sin(3x)$. (Chain Rule)
5) Find the derivative of the function $f(x)=x^5$. ((Simple) Power Rule)	6) Find the derivative of the function $f(x)=x^3-4x+7$. (Sum or Difference Rule)
7) Find the derivative of the function $f(x)=\dfrac{x^2}{\sin(x)}$. (Quotient Rule)	8) Find the derivative of the function $f(x)=(3x^2+2)^4$. (General Power Rule)
9) Find the derivative of the function $f(x)=\tan(x)$. (Derivatives of Trigonometric Functions)	10) Find the derivative of the function $f(x)=\sin(x)$. (Derivative of $\sin(x)$) -
11) Find the derivative of the function $f(x)=\cos(x)$. (Derivative of $\cos(x)$)	12) Find the derivative of the function $f(x)=\tan(2x)$. (Derivative of $\tan(x)$)
13) Find the derivative of the function $f(x)=\cot(x)$. (Derivative of $\cot(x)$)	14) Find the derivative of the function $f(x)=\sec(x)$. (Derivative of $\sec(x)$)

15) Find the derivative of the function $f(x)=\csc(x)$. (Derivative of $\csc(x)$)	16) Find the derivative of the function $f(x)=\sin(3x^2+2x)$. (Derivative of $\sin(u)$ where u is a function of x)
17) Find the derivative of the function $f(x)=\cos(x^3-x)$. (Derivative of $\cos(u)$ where u is a function of x)	18) Find the derivative of the function $f(x)=\tan(2x^2-3x)$. (Derivative of $\tan(u)$ where u is a function of x)
19) Find the derivative of the function $f(x)=\cot(4x^3+x^2)$. (Derivative of $\cot(u)$ where u is a function of x)	20) Find the derivative of the function $f(x)=\sec(3x^2+x)$. (Derivative of $\sec(u)$ where u is a function of x)
21) Find the derivative of the function $f(x)=\csc(x^2+2x)$. (Derivative of $\csc(u)$ where u is a function of x)	22) Find the derivative of the function $f(x)=\sinh(x)$. (Derivative of $\sinh(x)$)
23) Find the derivative of the function $f(x)=\cosh(x)$. (Derivative of $\cosh(x)$)	24) Find the derivative of the function $f(x)=\tanh(x)$. (Derivative of $\tanh(x)$)
25) Find the derivative of the function $f(x)=\text{sech}(x)$. (Derivative of $\text{sech}(x)$)	26) Find the derivative of the function $f(x)=\text{csch}(x)$. (Derivative of $\text{csch}(x)$)
27) Find the derivative of the function $f(x)=\coth(x)$. (Derivative of $\coth(x)$)	28) Find the derivative of the function $f(x)=\sinh^{-1}(x)$. (Derivative of $\sinh^{-1}(x)$)
29) Find the derivative of the function $f(x)=\cosh^{-1}(x)$. (Derivative of $\cosh^{-1}(x)$)	30) Find the derivative of the function $f(x)=\tanh^{-1}(x)$. (Derivative of $\tanh^{-1}(x)$)

31) Find the derivative of the function $f(x) = \text{sech}^{-1}(x)$. (Derivative of $\text{sech}^{-1}(x)$)	32) Find the derivative of the function $f(x) = \text{csch}^{-1}(x)$. (Derivative of $\text{csch}^{-1}(x)$)
33) Find the derivative of the function $f(x) = \coth^{-1}(x)$. (Derivative of $\coth^{-1}(x)$)	34) Find the derivative of the function $f(x) = e^x$. (Derivative of e^x)
35) Find the derivative of the function $f(x) = \ln(x)$. (Derivative of $\ln(x)$)	36) Find the derivative of the function $f(x) = 2^x$. (Derivative of a^x)
37) Find the derivative of the function $f(x) = \log_2(x)$. (Derivative of $\log_a(x)$)	38) Find the derivative of the function $f(x) = (3x^2 + 2)^5$. (Derivative of u^n)
39) Find the derivative of the function $f(x) = \|3x - 4\|$. (Derivative of $\|u\|$)	40) Find the derivative of the function $f(x) = \ln(2x^3 + 5)$. (Derivative of $\ln(u)$)
41) Find the derivative of the function $f(x) = e^{4x^2}$. (Derivative of e^u)	42) Find the derivative of the function $f(x) = \log_3(x^2 + 1)$. (Derivative of $\log_a(u)$)
43) Find the derivative of the function $f(x) = 5^{3x}$. (Derivative of a^u)	

Solutions:

1) Find the derivative of the function $f(x)=7$ **. (Constant Rule)** - Using the constant rule, which states $\frac{d}{dx}[c]=0$ - $f'(x)=\frac{d}{dx}[7]=0$	**2) Find the derivative of the function** $f(x)=5x^3$ **. (Constant Multiple Rule)** - Using the constant multiple rule, which states $\frac{d}{dx}[cu]=cu'$ - $f'(x)=\frac{d}{dx}[5x^3]=5\cdot\frac{d}{dx}[x^3]=5\cdot 3x^2=15x^2$
3) Find the derivative of the function $f(x)=x^2\sin(x)$ **. (Product Rule)** - Using the product rule, which states $\frac{d}{dx}[uv]=uv'+vu'$ - $u=x^2$, $v=\sin(x)$ - $u'=2x$, $v'=\cos(x)$ - $f'(x)=(x^2)'\sin(x)+x^2(\sin(x))'$ $=2x\sin(x)+x^2\cos(x)$	**4) Find the derivative of the function** $f(x)=\sin(3x)$ **. (Chain Rule)** - Using the chain rule, which states $\frac{d}{dx}[f(u)]=f'(u)u'$ - $f(u)=\sin(u)$, $u=3x$ - $f'(u)=\cos(u)$, $u'=3$ - $f'(x)=\cos(3x)\cdot 3=3\cos(3x)$
5) Find the derivative of the function $f(x)=x^5$ **. ((Simple) Power Rule)** - Using the power rule, which states $\frac{d}{dx}[x^n]=nx^{n-1}$ - $f'(x)=\frac{d}{dx}[x^5]=5x^4$	**6) Find the derivative of the function** $f(x)=x^3-4x+7$ **. (Sum or Difference Rule)** - Using the sum or difference rule, which states $\frac{d}{dx}[u\pm v]=u'\pm v'$ - $f'(x)=\frac{d}{dx}[x^3]-\frac{d}{dx}[-4x]+\frac{d}{dx}[7]$ $=3x^2-4+0=3x^2-4$
7) Find the derivative of the function $f(x)=\frac{x^2}{\sin(x)}$ **. (Quotient Rule)** - Using the quotient rule, which states $\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$ - $u=x^2$, $v=\sin(x)$ - $u'=2x$, $v'=\cos(x)$	**8) Find the derivative of the function** $f(x)=(3x^2+2)^4$ **. (General Power Rule)** - Using the general power rule, which states $\frac{d}{dx}[u^n]=nu^{n-1}u'$ - $u=3x^2+2$, $u'=6x$ - $n=4$ - $f'(x)=4(3x^2+2)^3\cdot 6x=24x(3x^2+2)^3$

- $f'(x) = \dfrac{\sin(x) \cdot 2x - x^2 \cdot \cos(x)}{\sin^2(x)}$ $= \dfrac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)}$	
9) Find the derivative of the function $f(x) = \tan(x)$. (Derivatives of Trigonometric Functions) - Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(x)] = \sec^2(x)$ - $f'(x) = \frac{d}{dx}[\tan(x)] = \sec^2(x)$	**10) Find the derivative of the function $f(x) = \sin(x)$. (Derivative of $\sin(x)$)** - Using the derivative rule for the sine function, which states $\frac{d}{dx}[\sin(x)] = \cos(x)$ - $f'(x) = \frac{d}{dx}[\sin(x)] = \cos(x)$
11) Find the derivative of the function $f(x) = \cos(x)$. (Derivative of $\cos(x)$) - Using the derivative rule for the cosine function, which states $\frac{d}{dx}[\cos(x)] = -\sin(x)$ - $f'(x) = \frac{d}{dx}[\cos(x)] = -\sin(x)$	**12) Find the derivative of the function $f(x) = \tan(2x)$. (Derivative of $\tan(x)$)** - Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(u)] = \sec^2(u) \cdot u'$ - $f'(x) = \frac{d}{dx}[\tan(2x)] = \sec^2(2x) \cdot 2$
13) Find the derivative of the function $f(x) = \cot(x)$. (Derivative of $\cot(x)$) - Using the derivative rule for the cotangent function, which states $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$ - $f'(x) = \frac{d}{dx}[\cot(x)] = -\csc^2(x)$	**14) Find the derivative of the function $f(x) = \sec(x)$. (Derivative of $\sec(x)$)** - Using the derivative rule for the secant function, which states $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ - $f'(x) = \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
15) Find the derivative of the function $f(x) = \csc(x)$. (Derivative of $\csc(x)$) - Using the derivative rule for the cosecant function, which states $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$ - $f'(x) = \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	**16) Find the derivative of the function $f(x) = \sin(3x^2 + 2x)$. (Derivative of $\sin(u)$ where u is a function of x)** - Using the chain rule and the derivative of sine, which states $\frac{d}{dx}[\sin(u)] = (\cos(u))u'$ - $u = 3x^2 + 2x$, $u' = 6x + 2$

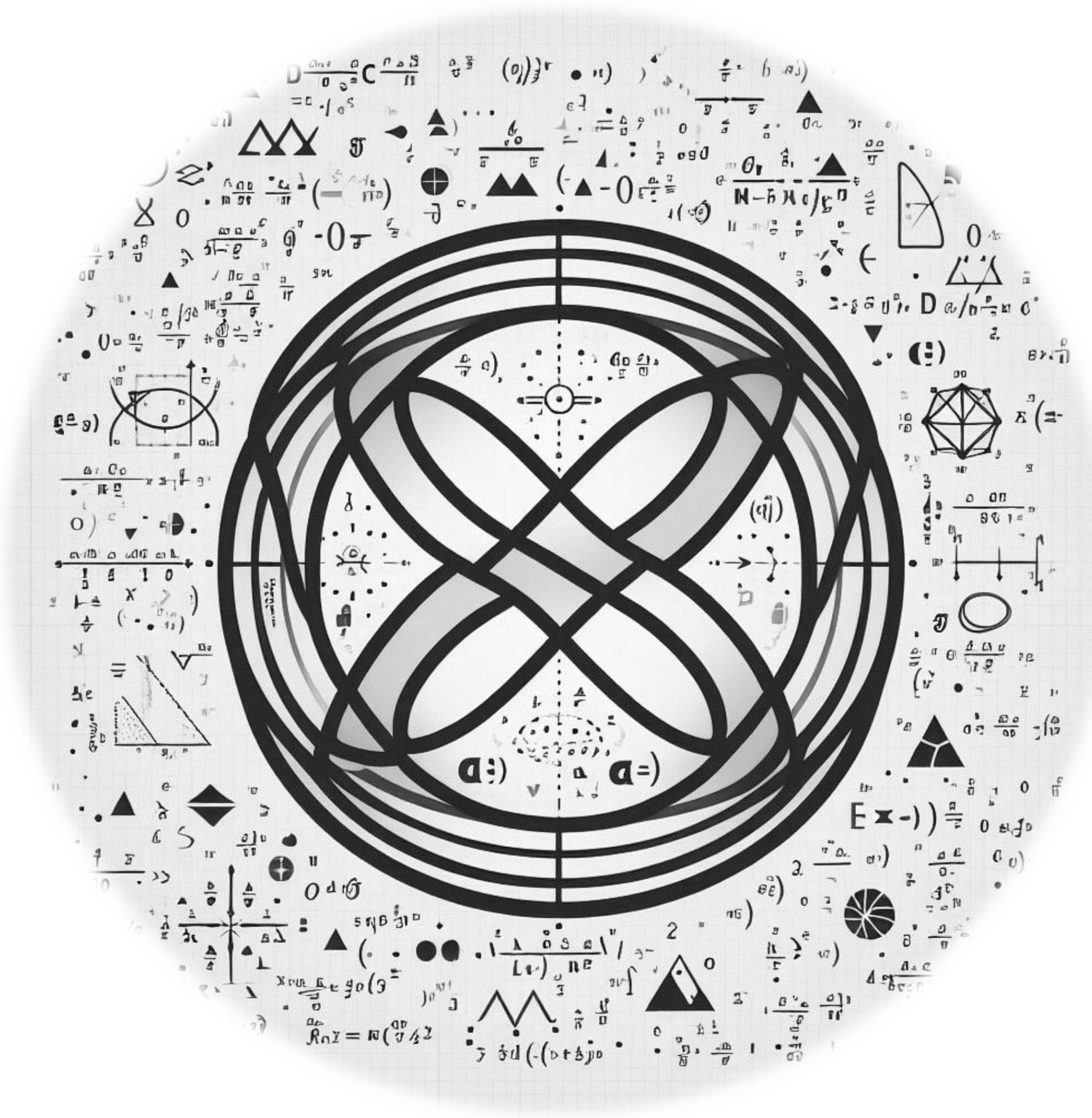
	- $f'(x) = \cos(3x^2 + 2x) \cdot (6x + 2)$ $= (6x + 2)\cos(3x^2 + 2x)$
17) Find the derivative of the function $f(x) = \cos(x^3 - x)$ **. (Derivative of** $\cos(u)$ **where u is a function of x)** - Using the chain rule and the derivative of cosine, which states $\frac{d}{dx}[\cos(u)] = -(\sin(u))u'$ - $u = x^3 - x$, $u' = 3x^2 - 1$ - $f'(x) = -\sin(x^3 - x) \cdot (3x^2 - 1)$ $= -(3x^2 - 1)\sin(x^3 - x)$	**18) Find the derivative of the function** $f(x) = \tan(2x^2 - 3x)$ **. (Derivative of** $\tan(u)$ **where u is a function of x)** - Using the chain rule and the derivative of tangent, which states $\frac{d}{dx}[\tan(u)] = (\sec^2(u))u'$ - $u = 2x^2 - 3x$, $u' = 4x - 3$ - $f'(x) = \sec^2(2x^2 - 3x) \cdot (4x - 3)$ $= (4x - 3)\sec^2(2x^2 - 3x)$
19) Find the derivative of the function $f(x) = \cot(4x^3 + x^2)$ **. (Derivative of** $\cot(u)$ **where u is a function of x)** - Using the chain rule and the derivative of cotangent, which states $\frac{d}{dx}[\cot(u)] = -(\csc^2(u))u'$ - $u = 4x^3 + x^2$, $u' = 12x^2 + 2x$ - $f'(x) = -\csc^2(4x^3 + x^2) \cdot (12x^2 + 2x)$ $= -(12x^2 + 2x)\csc^2(4x^3 + x^2)$	**20) Find the derivative of the function** $f(x) = \sec(3x^2 + x)$ **. (Derivative of** $\sec(u)$ **where u is a function of x)** - Using the chain rule and the derivative of secant, which states $\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))u'$ - $u = 3x^2 + x$, $u' = 6x + 1$ - $f'(x) = \sec(3x^2 + x)\tan(3x^2 + x) \cdot (6x + 1)$ $= (6x + 1)\sec(3x^2 + x)\tan(3x^2 + x)$
21) Find the derivative of the function $f(x) = \csc(x^2 + 2x)$ **. (Derivative of** $\csc(u)$ **where u is a function of x)** - Using the chain rule and the derivative of cosecant, which states $\frac{d}{dx}[\csc(u)] = -(\csc(u)\cot(u))u'$ - $u = x^2 + 2x$, $u' = 2x + 2$ - $f'(x) = -\csc(x^2 + 2x)\cot(x^2 + 2x) \cdot (2x + 2)$ $= -(2x + 2)\csc(x^2 + 2x)\cot(x^2 + 2x)$	**22) Find the derivative of the function** $f(x) = \sinh(x)$ **. (Derivative of** $\sinh(x)$ **)** - Using the derivative rule for the hyperbolic sine function, which states $\frac{d}{dx}[\sinh(x)] = \cosh(x)$ - $f'(x) = \frac{d}{dx}[\sinh(x)] = \cosh(x)$
23) Find the derivative of the function $f(x) = \cosh(x)$ **. (Derivative of** $\cosh(x)$ **)**	**24) Find the derivative of the function** $f(x) = \tanh(x)$ **. (Derivative of** $\tanh(x)$ **)**

- Using the derivative rule for the hyperbolic cosine function, which states $\frac{d}{dx}[\cosh(x)]=\sinh(x)$ - $f'(x)=\frac{d}{dx}[\cosh(x)]=\sinh(x)$	- Using the derivative rule for the hyperbolic tangent function, which states $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$ - $f'(x)=\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$
25) Find the derivative of the function $f(x)=\text{sech}(x)$ **. (Derivative of** $\text{sech}(x)$ **)** - Using the derivative rule for the hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$ - $f'(x)=\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$	**26) Find the derivative of the function** $f(x)=\text{csch}(x)$ **. (Derivative of** $\text{csch}(x)$ **)** - Using the derivative rule for the hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$ - $f'(x)=\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$
27) Find the derivative of the function $f(x)=\coth(x)$ **. (Derivative of** $\coth(x)$ **)** - Using the derivative rule for the hyperbolic cotangent function, which states $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$ - $f'(x)=\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$	**28) Find the derivative of the function** $f(x)=\sinh^{-1}(x)$ **. (Derivative of** $\sinh^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic sine function, which states $\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$ - $f'(x)=\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$
29) Find the derivative of the function $f(x)=\cosh^{-1}(x)$ **. (Derivative of** $\cosh^{-1}(x)$ **):** - Using the derivative rule for the inverse hyperbolic cosine function, which states $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$ - $f'(x)=\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$	**30) Find the derivative of the function** $f(x)=\tanh^{-1}(x)$ **. (Derivative of** $\tanh^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic tangent function, which states $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$ - $f'(x)=\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$
31) Find the derivative of the function $f(x)=\text{sech}^{-1}(x)$ **. (Derivative of** $\text{sech}^{-1}(x)$ **)**	**32) Find the derivative of the function** $f(x)=\text{csch}^{-1}(x)$ **. (Derivative of** $\text{csch}^{-1}(x)$ **)**

- Using the derivative rule for the inverse hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$ - $f'(x) = \frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$	- Using the derivative rule for the inverse hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$ - $f'(x) = \frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$
33) Find the derivative of the function $f(x) = \coth^{-1}(x)$**. (Derivative of** $\coth^{-1}(x)$**)** - Using the derivative rule for the inverse hyperbolic cotangent function, which states $\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$ - $f'(x) = \frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$	**34) Find the derivative of the function** $f(x) = e^x$**. (Derivative of** e^x**)** - Using the derivative rule for the exponential function, which states $\frac{d}{dx}[e^x] = e^x$ - $f'(x) = \frac{d}{dx}[e^x] = e^x$
35) Find the derivative of the function $f(x) = \ln(x)$**. (Derivative of** $\ln(x)$**)** - Using the derivative rule for the natural logarithm function, which states $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ - $f'(x) = \frac{d}{dx}[\ln(x)] = \frac{1}{x}$	**36) Find the derivative of the function** $f(x) = 2^x$**. (Derivative of** a^x**)** - Using the derivative rule for the exponential function with base a, which states $\frac{d}{dx}[a^x] = (\ln a)a^x$ - $f'(x) = \frac{d}{dx}[2^x] = (\ln 2)2^x$
37) Find the derivative of the function $f(x) = \log_2(x)$**. (Derivative of** $\log_a(x)$**)** - Using the derivative rule for the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(x)] = \frac{1}{(\ln a)x}$ - $f'(x) = \frac{d}{dx}[\log_2(x)] = \frac{1}{(\ln 2)x}$	**38) Find the derivative of the function** $f(x) = (3x^2+2)^5$**. (Derivative of** u^n**)** - Using the general power rule, which states $\frac{d}{dx}[u^n] = nu^{n-1}u'$ - $u = 3x^2+2$, $u' = 6x$ - $f'(x) = 5(3x^2+2)^4 \cdot 6x = 30x(3x^2+2)^4$
39) Find the derivative of the function $f(x) = \lvert 3x-4\rvert$**. (Derivative of |u|)**	**40) Find the derivative of the function** $f(x) = \ln(2x^3+5)$**. (Derivative of** $\ln(u)$**)**

- Using the rule for the derivative of the absolute value function, which states $\frac{d}{dx}[\lvert u\rvert]=\frac{u}{\lvert u\rvert}u'$ where $u\neq 0$ - $u=3x-4$, $u'=3$ - $f'(x)=\frac{3x-4}{\lvert 3x-4\rvert}\cdot 3=\frac{3(3x-4)}{\lvert 3x-4\rvert}$	- Using the rule for the derivative of the natural logarithm function, which states $\frac{d}{dx}[\ln(u)]=\frac{u'}{u}$ - $u=2x^3+5$, $u'=6x^2$ - $f'(x)=\frac{6x^2}{2x^3+5}$
41) Find the derivative of the function $f(x)=e^{4x^2}$. (Derivative of e^u) - Using the rule for the derivative of the exponential function, which states $\frac{d}{dx}[e^u]=e^u u'$ - $u=4x^2$, $u'=8x$ - $f'(x)=e^{4x^2}\cdot 8x=8xe^{4x^2}$	**42) Find the derivative of the function $f(x)=\log_3(x^2+1)$. (Derivative of $\log_a(u)$)** - Using the rule for the derivative of the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(u)]=\frac{u'}{(\ln a)u}$ - $u=x^2+1$, $u'=2x$ - $f'(x)=\frac{2x}{(\ln 3)(x^2+1)}=\frac{2x}{(\ln 3)(x^2+1)}$
43) Find the derivative of the function $f(x)=5^{3x}$. (Derivative of a^u) - Using the rule for the derivative of the exponential function with base a, which states $\frac{d}{dx}[a^u]=(\ln a)a^u u'$ - $u=3x$, $u'=3$ - $f'(x)=(\ln 5)5^{3x}\cdot 3=3(\ln 5)5^{3x}$	

Chapter 9. Differential Equations



9-1. Definitions and Basic Concepts of Differential Equations

Introduction to Differential Equations:

A differential equation is an equation that relates one or more functions and their derivatives. In applications, these functions usually represent physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between them.

Key Concepts:

1. **Order of a Differential Equation: The highest derivative present in the equation.**
2. **General Solution: A solution involving one or more arbitrary constants (C).**
3. **Particular Solution: A solution obtained from the general solution by specifying values of the constants so that certain conditions are satisfied.**
4. **Initial Value Problem (IVP): A differential equation together with a specified value, called an initial condition, which is a value of the function at a particular point.**

1) Solve the differential equation: $\frac{dy}{dx}=4x$	2) Solve the initial value problem (IVP): $\frac{dy}{dx}=e^x$ with the initial condition $y(0)=2$.
3) Find the general solution of the second-order differential equation: $\frac{d^2y}{dx^2}=6$	4) Solve the differential equation using separation of variables: $\frac{dy}{dx}=y^2\sin(x)$

5) Solve the initial value problem (IVP): $\frac{dy}{dx}=y(1-y)$ with the initial condition $y(0)=0.5$.	6) Solve the second-order differential equation: $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

Solutions:

1) Solve the differential equation: $\frac{dy}{dx}=4x$ To solve the differential equation, we integrate both sides with respect to x: $\int\frac{dy}{dx}dx=\int 4x\,dx$ $y=4\int x\,dx$ $y=4\left(\frac{x^2}{2}\right)+C$ $y=2x^2+C$ where C is the constant of integration.	**2) Solve the initial value problem (IVP):** $\frac{dy}{dx}=e^x$ **with the initial condition** $y(0)=2$**.** First, find the general solution by integrating both sides with respect to x: $\int\frac{dy}{dx}dx=\int e^x\,dx$ $y=e^x+C$ Now, use the initial condition $y(0)=2$: $2=e^0+C$ $2=1+C$ $C=1$ So, the particular solution is: $y=e^x+1$
3) Find the general solution of the second-order differential equation: $\frac{d^2y}{dx^2}=6$ First, integrate with respect to x to find the first derivative: $\int\frac{d^2y}{dx^2}dx=\int 6dx$ $\frac{dy}{dx}=6x+C_1$ Next, integrate again with respect to x to find y: $\int\frac{dy}{dx}dx=\int(6x+C_1)dx$ $y=6\int x\,dx+C_1\int 1dx$ $y=6\left(\frac{x^2}{2}\right)+C_1x+C_2=3x^2+C_1x+C_2$ where C_1 and C_2 are constants of integration.	**4) Solve the differential equation using separation of variables:** $\frac{dy}{dx}=y^2\sin(x)$ Separate the variables y and x: $\frac{1}{y^2}dy=\sin(x)dx$ Integrate both sides: $\int\frac{1}{y^2}dy=\int\sin(x)dx$ $-\frac{1}{y}=-\cos(x)+C$ Multiply through by -1: $\frac{1}{y}=\cos(x)-C$ Take the reciprocal: $y=\frac{1}{\cos(x)-C}$

5) Solve the initial value problem (IVP):	**6) Solve the second-order differential equation:**
$\frac{dy}{dx} = y(1-y)$ **with the initial condition** $y(0) = 0.5$. Separate the variables: $\frac{1}{y(1-y)}dy = dx$ Use partial fraction decomposition: $\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$ So the equation becomes: $\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = dx$ Integrate both sides: $\int\left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int dx$ $\ln\lvert y\rvert - \ln\lvert 1-y\rvert = x + C$ Combine logarithms: $\ln\left\lvert\frac{y}{1-y}\right\rvert = x + C$ Exponentiate both sides: $\frac{y}{1-y} = e^{x+C} = Ce^{x}$ Solve for y : $y = \frac{Ce^{x}}{1+Ce^{x}}$ Use the initial condition $y(0) = 0.5$: $0.5 = \frac{C}{1+C} \Rightarrow 0.5 + 0.5C = C \Rightarrow C = 1$ So the particular solution is: $y = \frac{e^{x}}{1+e^{x}}$	$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ First, solve the characteristic equation: $r^2 - 3r + 2 = 0$ Factor the quadratic equation: $(r-1)(r-2) = 0$ So, the roots are $r = 1$ and $r = 2$. Therefore, the general solution is: $y = C_1e^{x} + C_2e^{2x}$ where C_1 and C_2 are constants of integration.

9-2. Separable Differential Equation

Introduction to Separable Differential Equations:

A separable differential equation is a type of ordinary differential equation (ODE) where the variables can be separated on opposite sides of the equation in the form:

$$\frac{dy}{dx} = g(x)h(y)$$

This allows integration of each side independently, simplifying the solution process.

Simple Example: Differential Equation: $\frac{dy}{dx} = xy$

- Separate the variables: $\frac{1}{y}dy = xdx$
- Integrate both sides: $\ln|y| = \frac{x^2}{2} + C$

1) Solve the differential equation: $\frac{dy}{dx} = 3xy$	2) Solve the initial value problem (IVP): $\frac{dy}{dx} = 2y\cos(x)$ with the initial condition $y(0) = 1$.
3) Solve the differential equation: $\frac{dy}{dx} = \frac{y^2}{x}$	4) Solve the initial value problem (IVP): $\frac{dy}{dx} = \frac{x}{y+1}$ with the initial condition $y(1) = 0$.

5) Solve the initial value problem (IVP): $\frac{dy}{dx}=\frac{y-2}{x}$ with the initial condition $y(1)=3$.	6) Solve the differential equation: $\frac{dy}{dx}=\frac{x+1}{y-1}$

Solution:

1) Solve the differential equation: $\frac{dy}{dx}=3xy$ Separate the variables: $\frac{1}{y}dy=3xdx$ Integrate both sides: $\int\frac{1}{y}dy=\int 3xdx$ $\ln\|y\|=\frac{3x^2}{2}+C$ Exponentiate both sides: $\|y\|=e^{\frac{3x^2}{2}+C}$ $y=\pm e^C e^{\frac{3x^2}{2}}$ Since y can be positive or negative, we write the general solution as: Let $C_1=e^C: y=\pm C_1 e^{\frac{3x^2}{2}}$	**2) Solve the initial value problem (IVP):** $\frac{dy}{dx}=2y\cos(x)$ **with the initial condition** $y(0)=1$ Separate the variables: $\frac{1}{y}dy=2\cos(x)dx$ Integrate both sides: $\int\frac{1}{y}dy=\int 2\cos(x)dx$ $\ln\|y\|=2\sin(x)+C$ Exponentiate both sides: $\|y\|=e^{2\sin(x)+C}$ $y=\pm e^C e^{2\sin(x)}$ Let $C_1=e^C$: $y=C_1e^{2\sin(x)}$ Use the initial condition $y(0)=1: 1=C_1e^{2\sin(0)}$ $1=C_1e^0$ $C_1=1$ So, the particular solution is: $y=e^{2\sin(x)}$
3) Solve the differential equation: $\frac{dy}{dx}=\frac{y^2}{x}$ Separate the variables: $\frac{1}{y^2}dy=\frac{1}{x}dx$	4) **Solve the initial value problem (IVP):** $\frac{dy}{dx}=\frac{x}{y+1}$ **with the initial condition** $y(1)=0$. Separate the variables: $(y+1)dy=xdx$ Integrate both sides: $\int(y+1)dy=\int xdx$

Integrate both sides: $\int \frac{1}{y^2}dy = \int \frac{1}{x}dx$ $-\frac{1}{y} = \ln\|x\| + C$ Multiply through by -1: $\frac{1}{y} = -\ln\|x\| - C$ Take the reciprocal: $y = \frac{1}{-\ln\|x\| - C} = -\frac{1}{\ln\|x\| + C}$	$\frac{y^2}{2} + y = \frac{x^2}{2} + C$ Use the initial condition $y(1) = 0$: $\frac{0^2}{2} + 0 = \frac{1^2}{2} + C$ $0 = \frac{1}{2} + C$ $C = -\frac{1}{2}$ So, the particular solution is: $\frac{y^2}{2} + y = \frac{x^2}{2} - \frac{1}{2}$
5) Solve the initial value problem (IVP): $\frac{dy}{dx} = \frac{y-2}{x}$ **with the initial condition** $y(1) = 3$ **.** Separate the variables: $\frac{1}{y-2}dy = \frac{1}{x}dx$ Integrate both sides: $\int \frac{1}{y-2}dy = \int \frac{1}{x}dx$ $\ln\|y-2\| = \ln\|x\| + C$ Exponentiate both sides: $\|y-2\| = e^{\ln\|x\|+C}$ $\|y-2\| = e^C\|x\|$ Let $C_1 = e^C$: $y - 2 = \pm C_1 x$ Use the initial condition $y(1) = 3$: $3 - 2 = \pm C_1(1)$ $1 = \pm C_1$ Thus, $C_1 = 1$ or $C_1 = -1$. Considering both cases: $y - 2 = x$ or $y - 2 = -x$ So, the particular solutions are: $y = x + 2$ or $y = -x + 2$ Since $y(1) = 3$, the solution that fits is: $y = x + 2$	**6) Solve the differential equation:** $\frac{dy}{dx} = \frac{x+1}{y-1}$ Separate the variables: $(y-1)dy = (x+1)dx$ Integrate both sides: $\int (y-1)dy = \int (x+1)dx$ $\frac{y^2}{2} - y = \frac{x^2}{2} + x + C$ So, the solution is: $\frac{y^2}{2} - y = \frac{x^2}{2} + x + C$ Multiply through by 2 to simplify: $y^2 - 2y = x^2 + 2x + K$ (let 2C=K)

9-3. Euler's Method for Approximating Differential Equations

Introduction to Euler's Method:

Euler's Method is a numerical technique for approximating solutions to first-order ordinary differential equations (ODEs). It uses a series of linear approximations to model the behavior of the differential equation over discrete intervals.

Basic Concept of Euler's Method:

Given a differential equation of the form $\frac{dy}{dx} = f(x,y)$ with an initial condition $y(x_0) = y_0$, you approximate y at subsequent points using:

$$y_{n+1} = y_n + f(x_n, y_n) \cdot h$$

where h is the step size and y_{n+1} is the approximate value of y at $x_{n+1} = x_n + h$.

1) Use Euler's Method with a step size of $h = 0.5$ to find the first approximation y_1 of the differential equation $\frac{dy}{dx} = 3y$ given that $y(0) = 1$.	2) Apply Euler's Method with $h = 0.2$ to estimate y_1 for $\frac{dy}{dx} = y - x$ starting from $x_0 = 0$, $y_0 = 2$.

3) Use Euler's Method with a step size of $h=0.1$ to approximate y at $x=0.2$ for the equation $\frac{dy}{dx}=y+e^x$ with $y(0)=0$.	4) Estimate $y(0.3)$ using Euler's Method with $h=0.1$ for $\frac{dy}{dx}=x^2+y^2$, starting from $y(0)=1$.
5) Apply Euler's Method to approximate the solution from $x=0$ to $x=1$ using $h=0.2$ for $\frac{dy}{dx}=\sin(x)+y$ with $y(0)=0$.	6) Use Euler's Method to solve $\frac{dy}{dx}=xy$ over [0, 0.5] with $h=0.1$ and $y(0)=1$.

Solutions:

1) Use Euler's Method with a step size of $h=0.5$ **to find the first approximation** y_1 **of the differential equation** $\frac{dy}{dx}=3y$ **given that** $y(0)=1$**.** Euler's Method formula: $y_{n+1}=y_n+hf(x_n,y_n)$ Given: $h=0.5$, $\frac{dy}{dx}=3y$, $y(0)=1$ We need to calculate: $y(0.5)$ Step 1: $x_0=0$, $y_0=1$ $y_1=y_0+h\cdot f(x_0,y_0)=1+0.5\cdot 3\cdot 1=2.5$ So, the first approximation at $x=0.5$ is $y(0.5)\approx 2.5$.	**2) Apply Euler's Method with** $h=0.2$ **to estimate** y_1 **for** $\frac{dy}{dx}=y-x$ **starting from** $x_0=0$**,** $y_0=2$**.** Euler's Method formula: $y_{n+1}=y_n+hf(x_n,y_n)$ Given: $h=0.2$, $\frac{dy}{dx}=y-x$, $x_0=0$, $y_0=2$ We need to calculate: $y(0.2)$ Step 1: $x_0=0$, $y_0=2$ $y_1=y_0+h\cdot f(x_0,y_0)=2+0.2\cdot(2-0)=2.4$ So, the first approximation at $x=0.2$ is $y(0.2)\approx 2.4$.
3) Use Euler's Method with a step size of $h=0.1$ **to approximate y at** $x=0.2$ **for the equation** $\frac{dy}{dx}=y+e^x$ **with** $y(0)=0$**.** 1. Initial values: $x_0=0$, $y_0=0$, $h=0.1$ 2. Euler's formula: $y_{n+1}=y_n+h\cdot f(x_n,y_n)$ where $f(x,y)=y+e^x$. 3. Calculate successive points: - For $n=0$: $y_1=y_0+h\cdot(y_0+e^{x_0})=0+0.1\cdot(0+e^0)=0.1$ $x_1=x_0+h=0+0.1=0.1$ - For $n=1$: $y_2=y_1+h\cdot(y_1+e^{x_1})$ $=0.1+0.1\cdot(0.1+e^{0.1})\approx 0.2205$ $x_2=x_1+h=0.1+0.1=0.2$ 4. Result: The approximations are: $y_1\approx 0.1$, $y_2\approx 0.2205$	**4) Estimate** $y(0.3)$ **using Euler's Method with** $h=0.1$ **for** $\frac{dy}{dx}=x^2+y^2$**, starting from** $y(0)=1$**.** 1. Initial values: $x_0=0$, $y_0=1$, $h=0.1$ 2. Euler's formula: $y_{n+1}=y_n+h\cdot f(x_n,y_n)$ where $f(x,y)=x^2+y^2$. 3. Perform three iterations: - For $n=0$: $y_1=y_0+h\cdot(x_0^2+y_0^2)=1+0.1\cdot(0^2+1^2)=1.1$ $x_1=x_0+h=0+0.1=0.1$ - For $n=1$: $y_2=y_1+h\cdot(x_1^2+y_1^2)$ $=1.1+0.1\cdot(0.1^2+1.1^2)=1.222$ $x_2=x_1+h=0.1+0.1=0.2$ - For $n=2$: $y_3=y_2+h\cdot(x_2^2+y_2^2)$ $=1.222+0.1\cdot(0.2^2+1.222^2)=1.3752884$ $x_3=x_2+h=0.2+0.1=0.3$ 4. Result: The approximations are: $y_1\approx 1.1$, $y_2\approx 1.222$ $y_3\approx 1.375$ Therefore, the approximate value of y at $x=0.3$ is $y_3\approx 1.375$.

5) Apply Euler's Method to approximate the solution from $x=0$ to $x=1$ using $h=0.2$ for $\frac{dy}{dx}=\sin(x)+y$ with $y(0)=0$.

Euler's Method formula: $y_{n+1}=y_n+hf(x_n,y_n)$

Given: $h=0.2$, $\frac{dy}{dx}=\sin(x)+y$, $y(0)=0$

We need to calculate:
$y(0.2), y(0.4), y(0.6), y(0.8), y(1.0)$

Step 1: $x_0=0$, $y_0=0$
$y_1=y_0+h(\sin(x_0)+y_0)=0+0.2(\sin(0)+0)$
$=0$
Step 2: $x_1=0.2$, $y_1=0$
$y_2=y_1+h(\sin(x_1)+y_1)=0+0.2(\sin(0.2)+0)$
$=0.03974$
Step 3: $x_2=0.4$, $y_2=0.03974$
$y_3=y_2+h(\sin(x_2)+y_2)$
$=0.03974+0.2(\sin(0.4)+0.03974)=0.12557$
Step 4: $x_3=0.6$, $y_3=0.12557$
$y_4=y_3+h(\sin(x_3)+y_3)$
$=0.12557+0.2(\sin(0.6)+0.12557)=0.26360$
Step 5: $x_4=0.8$, $y_4=0.26360$
$y_5=y_4+h(\sin(x_4)+y_4)$
$=0.26360+0.2(\sin(0.8)+0.26360)=0.45980$
So, the approximate solution at $x=1$ is $y(1)\approx 0.45980$.

6) Use Euler's Method to solve $\frac{dy}{dx}=xy$ over [0, 0.5] with $h=0.1$ and $y(0)=1$.

Euler's Method formula: $y_{n+1}=y_n+hf(x_n,y_n)$

Given: $h=0.1$, $\frac{dy}{dx}=xy$, $y(0)=1$

We need to calculate:
$y(0.1), y(0.2), y(0.3), y(0.4), y(0.5)$

Step 1: $x_0=0$, $y_0=1$
$y_1=y_0+h(x_0y_0)=1+0.1(0\cdot 1)=1$
Step 2: $x_1=0.1$, $y_1=1$
$y_2=y_1+h(x_1y_1)=1+0.1(0.1\cdot 1)=1.01$
Step 3: $x_2=0.2$, $y_2=1.01$
$y_3=y_2+h(x_2y_2)=1.01+0.1(0.2\cdot 1.01)$
$=1.0302$
Step 4: $x_3=0.3$, $y_3=1.0302$
$y_4=y_3+h(x_3y_3)=1.0302+0.1(0.3\cdot 1.0302)$
$=1.061106$
Step 5: $x_4=0.4$, $y_4=1.061106$
$y_5=y_4+h(x_4y_4)$
$=1.061106+0.1(0.4\cdot 1.061106)$
$=1.10355024$
So, the approximate solution at $x=0.5$ is $y(0.5)\approx 1.1036$.

9-4. Exponential Growth and Decay Model by Differential Equations

Introduction to Exponential Growth and Decay:

Exponential growth and decay are processes that increase or decrease at rates proportional to their current value. These are commonly modeled with differential equations of the form:

$$\frac{dy}{dt} = ky$$

where y is the quantity of interest, T is time, and k is a constant. For $k > 0$, the equation models growth, and for $k < 0$, it models decay.

1) Solve the differential equation for a population experiencing exponential growth at a rate of 20% per year with an initial population of 500.	2) A radioactive substance decays at a rate proportional to its current amount with a decay constant of -0.05 per day. Write the differential equation for the decay process.
3) Determine the half-life of a radioactive material if the decay rate is -0.693 per year.	4) A culture of bacteria doubles every 3 hours. If you start with 100 bacteria, how many bacteria will there be after 24 hours?

5) A drug is administered to a patient and decays exponentially in the bloodstream with a decay rate of 30% per hour. If the initial dose was 200 mg, how much of the drug remains after 5 hours?	6) A cooling body follows Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - T_{\text{env}})$, where T is the temperature of the body, T_{env} is the environmental temperature, and k is a positive constant. If a body cools from 100°C to 70°C in 10 minutes in a room at 20°C, find the temperature of the body after another 10 minutes.

Solutions:

1) Solve the differential equation for a population experiencing exponential growth at a rate of 20% per year with an initial population of 500. - Given the equation $\frac{dP}{dt} = 0.2P$, solving with $P(0) = 500$: - $P(t) = 500e^{0.2t}$	**2) A radioactive substance decays at a rate proportional to its current amount with a decay constant of -0.05 per day. Write the differential equation for the decay process.** - The equation is $\frac{dA}{dt} = -0.05A$, where A is the amount of the substance.
3) Determine the half-life of a radioactive material if the decay rate is -0.693 per year. - $T_{1/2} = \frac{\ln 2}{\lvert k \rvert} = \frac{\ln 2}{0.693}$ - Half-life T is found from $e^{-0.693T} = \frac{1}{2}$, giving $T = 1$ year.	**4) A culture of bacteria doubles every 3 hours. If you start with 100 bacteria, how many bacteria will there be after 24 hours?** - Doubling time of 3 hours implies a growth rate k where $e^{3k} = 2$. - Thus, $k = \frac{\ln 2}{3}$. The model is: $N(24) = 100e^{\left(\frac{\ln 2}{3}\right)\cdot k}$ - After 24 hours ($t = 24$): $N(24) = 100e^{\left(\frac{\ln 2}{3}\right)\cdot 24}$ - $N(24) = 100e^{8\ln 2} = 25600$ So, there will be 25,600 bacteria after 24 hours.

5) A drug is administered to a patient and decays exponentially in the bloodstream with a decay rate of 30% per hour. If the initial dose was 200 mg, how much of the drug remains after 5 hours? - The equation is $\frac{dD}{dt} = -0.3D$. - The solution with $D(0) = 200$ is: $D(t) = 200e^{-0.3t}$ - After 5 hours: $D(5) = 200e^{-1.5} \approx 200 \times 0.223 = 44.6$ mg	**6) A cooling body follows Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - T_{env})$, where T is the temperature of the body, T_{env} is the environmental temperature, and k is a positive constant. If a body cools from 100°C to 70°C in 10 minutes in a room at 20°C, find the temperature of the body after another 10 minutes.** - Solve differential equation: $\int \frac{1}{T - T_{env}} dT = -k \int dt$ - First, find k using the initial condition and solve: $T(t) = T_{env} + (T_0 - T_{env})e^{-kt}$ - $T(t) - 20 = (100 - 20)e^{-kt}$ - Given $T(0) = 100$, $T(10) = 70$, and $T_{env} = 20$, set up the equation and solve for k, then use it to find $T(20)$ - $70 - 20 = 80e^{-10k} \Rightarrow k = -\frac{1}{10}\ln\left(\frac{5}{8}\right)$ - $T(20) - 20 = 80e^{-k \cdot 20} \Rightarrow T(20) = 51.25^{\circ}$C

9-5. Radioactive Decay Model by Differential Equations

Introduction to Radioactive Decay:

Radioactive decay is a natural process by which an unstable atomic nucleus loses energy by emitting radiation. The rate of decay is proportional to the current amount of the substance, which leads to the mathematical model:

$$\frac{dN}{dt} = -kN$$

where N is the amount of the radioactive substance, T is time, and k is the positive decay constant.

1) Write the differential equation representing the decay of a substance at a rate of 5% per year. If you start with 200 grams, what is the initial condition?	2) Given the radioactive decay model $\frac{dN}{dt} = -0.04N$ and an initial amount of 150 grams, find the amount of the substance remaining after 10 years.
3) A radioactive substance has a half-life of 6 years. Write the differential equation for its decay, and determine the decay constant k.	4) Determine the time it takes for a substance to decay from 1000 grams to 100 grams if the decay constant k is 0.123.

5) Model the radioactive decay of a substance with a half-life of 3 years, starting from an initial mass of 500 grams. Calculate how much of the substance remains after 15 years.	6) If the activity (rate of decay) of a radioactive substance is initially measured at 400 Bq (becquerels) and decreases to 50 Bq over a period of 10 years, determine the decay constant and the initial amount of the substance.

Solutions:

1) Write the differential equation representing the decay of a substance at a rate of 5% per year. If you start with 200 grams, what is the initial condition? - The differential equation is $\frac{dN}{dt} = -0.05N$. - The initial condition is $N(0) = 200$ grams.	**2) Given the radioactive decay model $\frac{dN}{dt} = -0.04N$ and an initial amount of 150 grams, find the amount of the substance remaining after 10 years.** - This is a first-order linear differential equation, and its solution is: $N(t) = N_0 e^{-kt}$ Given: - k = 0.04 - Initial amount $N_0 = 150$ grams - Time $t = 10$ years - Solve the equation $N(t) = 150e^{-0.04t}$. After 10 years: $N(10) \approx 150 \cdot 0.67032$ $N(10) \approx 100.548$grams
3) A radioactive substance has a half-life of 6 years. Write the differential equation for its decay, and determine the decay constant k.	**4) Determine the time it takes for a substance to decay from 1000 grams to 100 grams if the decay constant k is 0.123.** - Using $N(t) = 1000e^{-0.123t}$, solve for T when $N(t) = 100$:

- Using the half-life formula $N(t)=N_0\times\left(\frac{1}{2}\right)^{\frac{t}{T}}$ where T is the half-life: $\ln\left(\frac{1}{2}\right)=-k\times 6\Rightarrow k=\frac{\ln(2)}{6}$ - The differential equation is $\frac{dN}{dt}=-\left(\frac{\ln(2)}{6}\right)N$.	$100=1000e^{-0.123t}\Rightarrow e^{-0.123t}=0.1$ $\Rightarrow t=\frac{\ln(0.1)}{-0.123}\approx 18.71$ years
5) Model the radioactive decay of a substance with a half-life of 3 years, starting from an initial mass of 500 grams. Calculate how much of the substance remains after 15 years. - Calculate k as $k=\frac{\ln(2)}{3}$, and use $N(t)=500e^{-\left(\frac{\ln(2)}{3}\right)t}$: $N(15)=500\times e^{-5\ln(2)}=500\times\left(\frac{1}{32}\right)$ ≈ 15.625 grams	**6) If the activity (rate of decay) of a radioactive substance is initially measured at 400 Bq (becquerels) and decreases to 50 Bq over a period of 10 years, determine the decay constant and the initial amount of the substance.** - Activity $A(t)=kN(t)$ where $A(t)=A_0e^{-kt}$. - Given $A(0)=400$ Bq and $A(10)=50$ Bq, solve for k: $50=400e^{-k\cdot 10}$ $\Rightarrow k=-\frac{\ln(0.125)}{10}\approx 0.208$ - Calculate k and then use $N(0)=\frac{A(0)}{k}$ to find the initial amount of the substance: $N_0=\frac{400}{0.208}\approx 1923.08\text{grams}$

9-6. Newton's Law of Cooling Model by Differential Equations

Introduction to Newton's Law of Cooling:

Newton's Law of Cooling states that the rate of change of the temperature T of an object is proportional to the difference between the temperature of the object and the ambient temperature T_{env}, expressed as:

$$\frac{dT}{dt} = -k(T - T_{\text{env}})$$

where k is the positive constant of proportionality.

1) Write the differential equation for an object heating up from 50°C in a room where the temperature is 70°C. The constant of proportionality k is 0.1.	2) Given an initial object temperature of 100°C and an ambient temperature of 20°C with $k = 0.05$, find the temperature of the object after 10 minutes.
3) An object with an initial temperature of 30°C is placed in a freezer at -10°C. If the cooling constant is 0.03, calculate the temperature of the object after 1 hour.	4) Determine how long it will take for an object at 85°C to cool to 25°C in an environment at 20°C with a cooling constant $k = 0.09$.
5) A cup of coffee at 90°C is placed in a room at 20°C. The constant k is 0.15. How long will it take for the coffee to cool to 50°C?	6) Model the cooling process of a metal rod from 150°C to a stable temperature of 25°C in a cooler at 25°C, and determine the cooling constant k if it reaches 100°C after 5 minutes.

Solutions:

1) Write the differential equation for an object heating up from 50°C in a room where the temperature is 70°C. The constant of proportionality k is 0.1. - The differential equation is $\frac{dT}{dt} = -0.1(T-70)$. - The negative sign indicates cooling, but since the object's temperature is below the ambient, it will actually warm up.	**2) Given an initial object temperature of 100°C and an ambient temperature of 20°C with $k = 0.05$, find the temperature of the object after 10 minutes.** - The differential equation is: $\frac{dT}{dt} = 0.05(20-T)$ - Use the law of cooling: $T(t) = 20 + (100-20)e^{-0.05 \cdot t}$ - At $t = 10$ minutes: $T(10) = 20 + 80e^{-0.5}$ $T(10) \approx 68.52^{\circ}\text{C}$
3) An object with an initial temperature of 30°C is placed in a freezer at -10°C. If the cooling constant is 0.03, calculate the temperature of the object after 1 hour. - The differential equation is: $\frac{dT}{dt} = 0.03(-10-T)$ - $T(t) = -10 + (30+10)e^{-0.03 \cdot t}$ - At $t = 60$ minutes: $T(60) = -10 + 40e^{-1.8}$ $T(60) \approx -3.388^{\circ}\text{C}$	**4) Determine how long it will take for an object at 85°C to cool to 25°C in an environment at 20°C with a cooling constant $k = 0.09$.** - The differential equation is: $\frac{dT}{dt} = 0.09(20-T)$ - Set up the equation: $T(t) = 20 + (85-20)e^{-0.09t}$ - Solve $25 = 20 + (85-20)e^{-0.09t}$ for t: $t \approx 28.5$ minutes So, it will take approximately 28.5 minutes for the object to cool from 85°C to 25°C.
5) A cup of coffee at 90°C is placed in a room at 20°C. The constant k is 0.15. How long will it take for the coffee to cool to 50°C? - The differential equation is: $\frac{dT}{dt} = 0.15(20-T)$ - $T(t) = 20 + (90-20)e^{-0.15t}$ - Set $T(t) = 50 : 50 = 20 + (90-20)e^{-0.15t}$ $t \approx 5.65$ minutes So, it will take approximately 5.65 minutes for the coffee to cool to 50°C.	**6) Model the cooling process of a metal rod from 150°C to a stable temperature of 25°C in a cooler at 25°C and determine the cooling constant k if it reaches 100°C after 5 minutes.** - The differential equation is: $\frac{dT}{dt} = k(25-T)$ - $T(5) = 25 + (150-25)e^{-5k} = 100$ - Solve for k: $k \approx 0.1022$ So, the cooling constant k is approximately 0.1022.